ETH Zürich
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Introduction to Mathematical Finance

Exercise sheet 1

Exercise 1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable with $X \geq 0$ P-a.s.. Prove that $E[X] = 0$ implies that $X = 0$ P-a.s..

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be the probability space with $\Omega := \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} := 2^\Omega$ and P defined by $P[\{\omega_1\}] := 0.5$ and $P[\{\omega_3\}] := 0.2$. Let $X : \Omega \to \mathbb{R}$ be the random variable defined by $X(\omega_1) := 12$, $X(\omega_2) := -4$ and $X(\omega_3) := -8$ and $\mathcal{G} := \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}$ a sub-σ-field of $\mathcal{F}$.

(a) What is $P[\{\omega_2\}]$?

(b) Think of X as the price change over one week of a Facebook share. Give a verbal description of the set $A := \{\omega_2, \omega_3\}$ in terms of X.

(c) Calculate the conditional expectation $E[X \mid \mathcal{G}]$.

(d) Find all probability measures Q that are equivalent to P on $\mathcal{F}$ and satisfy

$$E_Q[X] = 4\,,$$

where $E_Q[\bullet]$ denotes the expectation under Q.

Exercise 1.3 Black Scholes closed formula
Let U_i iid $\sim N(0,1)$ and

$$S_n = e^{\left(r - \frac{\sigma^2}{2}\right) n + \sigma \sum_{i=1}^{n} U_i}$$

for $0 \leq n \leq N$. Given the filtration $\mathcal{F}_n = \sigma(S_0, \ldots, S_n)$, show that

$$E[e^{-r(N-n)}(S_N - K)_+ | \mathcal{F}_n] = S_n N(d_1) - K e^{-r(N-n)} N(d_2)$$

where $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$, is the cumulative distribution function of a standard normal variable.

$$d_1 = \frac{\ln\left(\frac{S_n}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\theta}{\sigma\sqrt{\theta}}$$
$$d_2 = d_1 - \sigma\sqrt{\theta}$$

where $\theta = N - n$. This is the so-called Black Scholes formula for a call option. The price of an European call option at time n is $Call(N-n, S_n, K, r, \sigma) = E[e^{-r(N-n)} f(S_N) | \mathcal{F}_n]$, the payoff of a call option is $f(S_N) = (S_N - K)_+$.

Exercise 1.4 Python - Black Scholes closed formula Write the function that compute the Black Scholes closed formula of a call option where (maturity, spot, strike, rate, vol) = ($N-n$, S_n, K, r, σ).

```python
from math import exp, log, sqrt
from scipy.stats import norm


def call_closed_formula(maturity, spot, strike, rate, vol):
  # TODO: implement.
  return 0
```