Introduction to Mathematical Finance

Solution sheet 1

Solution 1.1 Since $X \geq 0$ P-a.s., it is sufficient to show that $P[X > 0] = 0$. Since for all $n \in \mathbb{N}$ we have the inequality

$$\mathbb{1}_{\{X \geq \frac{1}{n}\}} \leq nX\,,$$

we conclude by taking on both sides expectation that

$$P\left[X \geq \frac{1}{n}\right] \leq nE[X] = 0$$

by assumption. But since $\{X > 0\}$ is the increasing union of all $\{X \geq \frac{1}{n}\}$, $n \in \mathbb{N}$, i.e., $\{X > 0\} = \cup_{n=1}^{\infty}\{X \geq \frac{1}{n}\}$ with $\mathbb{1}_{\{X>0\}} = \lim_{n\to\infty} \mathbb{1}_{\{X \geq \frac{1}{n}\}}$ P-a.s., the monotone convergence theorem yields that

$$P[X > 0] = \lim_{n\to\infty} P\left[X \geq \frac{1}{n}\right] = 0\,.$$

Solution 1.2

(a) By definition of a probability measure, we must have $P[\Omega] = 1$. Thus, $P[\{\omega_2\}] = 1 - P[\{\omega_1\}] - P[\{\omega_3\}] = 0.3$.

(b) $A = \{\omega_2, \omega_3\} = \{X = -4 \text{ or } X = -8\}$ is the event that the share becomes cheaper.

(c) Denote $A := \{\omega_1\}$. Then $\mathcal{G} = \sigma(A, A^c)$. This means that $\mathcal{G}$ is generated by two disjoint sets. Hence the conditional expectation takes the form

$$E[X \,|\, \mathcal{G}] = \frac{E[X\mathbb{1}_A]}{P[A]}\mathbb{1}_A + \frac{E[X\mathbb{1}_{A^c}]}{P[A^c]}\mathbb{1}_{A^c}\,.$$

Since $\frac{E[X\mathbb{1}_A]}{P[A]} = 12$ and

$$\frac{E[X\mathbb{1}_{A^c}]}{P[A^c]} = \frac{1}{0.5}\left(-4P[\{\omega_2\}] - 8P[\{\omega_3\}]\right) = -5.6\,,$$

we obtain the explicit representation

$$E[X \,|\, \mathcal{G}] = 12 \times \mathbb{1}_A - 5.6 \times \mathbb{1}_{A^c} = \begin{cases} 12 & \text{on } \{\omega_1\}, \\ -5.6 & \text{on } \{\omega_2, \omega_3\}. \end{cases}.$$

(d) We first note that $Q \approx P$ if and only if $q_k := Q[\{\omega_k\}] > 0$ for $k = 1, 2, 3$. Thus an equivalent probability measure can be identified with a triplet (q_1, q_2, q_3) of strictly positive real numbers whose sum is one. Computing the Q-expectation yields

$$E_Q[X] = 12q_1 - 4q_2 - 8q_3\,.$$

Thus, the set of equivalent probability measures Q for which $4 = E_Q[X]$ holds can be represented by the solution of the linear system

$$\begin{cases} 3q_1 - q_2 - 2q_3 - 1 & = 0, \\ q_1 + q_2 + q_3 - 1 & = 0, \\ q_1, q_2, q_3 & > 0. \end{cases}$$

The solution to this is given by

$$\begin{cases} q_1 &= \frac{1}{4}(\lambda+2), \\ q_2 &= \frac{1}{4}(2-5\lambda), \\ q_3 &= \lambda \in \left(0, \frac{2}{5}\right). \end{cases}$$

Solution 1.3 Black Scholes closed formula

$$\mathbb{E}[e^{-r(N-n)}(S_N-K)_+|\mathcal{F}_n] = \mathbb{E}\left[e^{-r(N-n)}S_n\left(\frac{S_N}{S_n}-\frac{K}{S_n}\right)_+\middle|\mathcal{F}_n\right]$$

As $\frac{S_N}{S_n} = e^{\left(r-\frac{\sigma^2}{2}\right)(N-n)+\sigma\sum_{i=n+1}^N U_i}$ is independent of $\mathcal{F}_n$, we get :

$$\mathbb{E}\left[e^{-r(N-n)}S_n\left(\frac{S_N}{S_n}-\frac{K}{S_n}\right)_+\middle|\mathcal{F}_n\right] = \varphi(S_n)$$

where

$$\begin{aligned}\varphi(x) &= \mathbb{E}\left[e^{-r(N-n)}x\left(\frac{S_N}{S_n}-\frac{K}{x}\right)_+\right] \\ &= \mathbb{E}\left[e^{-r(N-n)}\left(x\frac{S_N}{S_n}-K\right)_+\right]\end{aligned}$$

We replace $\frac{S_N}{S_n}$ and we get:

$$\varphi(x) = \mathbb{E}\left[e^{-r(N-n)}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)(N-n)+\sigma\sum_{i=n+1}^N U_i}-K\right)_+\right]$$

As U_i are iid $\sim N(0,1)$, then $\sum_{i=n+1}^N U_i \sim N(0, N-n)$.

$$\varphi(x) = \mathbb{E}\left[e^{-r(N-n)}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)(N-n)+\sigma\sqrt{N-n}\ U_1}-K\right)_+\right]$$

Then we will use $(a-b)_+ = (a-b)1_{a>b}$. Let call $\theta = N-n$.

$$\begin{aligned} xe^{\left(r-\frac{\sigma^2}{2}\right)\theta+\sigma\sqrt{\theta}\ U_1} > K &\Leftrightarrow \left(r-\frac{\sigma^2}{2}\right)\theta+\sigma\sqrt{\theta}U_1 > \log\left(\frac{K}{x}\right) \\ &\Leftrightarrow \sigma\sqrt{\theta}U_1 > \log\left(\frac{K}{x}\right)-\left(r-\frac{\sigma^2}{2}\right)\theta \\ &\Leftrightarrow U_1 > -\frac{\log\left(\frac{x}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} := -d_2 \end{aligned}$$

So we have:

$$\begin{aligned}\varphi(x) &= \mathbb{E}\left[e^{-r\theta}\left(xe^{\left(r-\frac{\sigma^2}{2}\right)\theta+\sigma\sqrt{\theta}U_1}-K\right)1_{(U_1>-d_2)}\right] \\ &= \mathbb{E}\left[xe^{-\frac{\sigma^2}{2}\theta+\sigma\sqrt{\theta}U_1}\ 1_{(U_1>-d_2)}-e^{-r\theta}K1_{(U_1>-d_2)}\right] \\ &= x\mathbb{E}\left[e^{-\frac{\sigma^2}{2}\theta}e^{\sigma\sqrt{\theta}U_1}\ 1_{(U_1>-d_2)}\right]-e^{-r\theta}K\ \mathbb{P}[U_1>-d_2]\end{aligned}$$

The first term can be written:

$$\begin{aligned}
\mathbb{E}\left[e^{-\frac{\sigma^2}{2}\theta}e^{\sigma\sqrt{\theta}U_1}1_{(U_1>-d_2)}\right] &= \int_{\mathbb{R}} e^{-\frac{\sigma^2}{2}\theta}e^{\sigma\sqrt{\theta}y}1_{y>-d_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
&= \int_{\mathbb{R}} 1_{(y>-d_2)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\sigma\sqrt{\theta})^2}{2}}dy \\
&= \int_{\mathbb{R}} 1_{\left(z>-(d_2+\sigma\sqrt{\theta})\right)}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= \mathbb{P}[U_1 > -(d_2+\sigma\sqrt{\theta})] = \mathbb{P}[U_1 > -d_1]
\end{aligned}$$

As $U_1 \sim N(0,1)$, $\mathbb{P}[U_1 > -d_1] = \mathbb{P}[U_1 \leq d_1]$, we have:

$$\begin{aligned}
\varphi(x) &= x\mathbb{P}[U_1 > -d_1] - e^{-r\theta}K\mathbb{P}[U_1 > -d_2] \\
&= x\mathbb{P}[U_1 \leq d_2 + \sqrt{\theta}] - e^{-r\theta}K\mathbb{P}[U_1 \leq d_2] \\
&= xN(d_1) - e^{-r\theta}KN(d_2)
\end{aligned}$$

Finally

$$\begin{aligned}
\mathbb{E}[e^{-r(N-n)}(S_N-K)_+|\mathcal{F}_n] &= \varphi(S_n) \\
&= S_nN(d_1) - e^{-r\theta}KN(d_2)
\end{aligned}$$

Solution 1.4

```python
from math import exp, log, sqrt
from scipy.stats import norm


def call_closed_formula(maturity, spot, strike, rate, vol):
  """Closed Formula for a call in Black Scholes.
  """
  sigmasqrt = vol * sqrt(maturity)
  d1 = (log(spot / strike) + rate * maturity) / sigmasqrt + sigmasqrt * 0.5
  d2 = d1 - sigmasqrt
  return spot * norm.cdf(d1) - exp(-rate * maturity) * strike * norm.cdf(d2)
```