ETH Zürich
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Introduction to Mathematical Finance

Exercise sheet 2

Exercise 2.1 Functional analysis. Let $(\Omega, \mathcal{F}, \mu)$ be a normed space. Let $1 \leq p < \infty$

$$\mathcal{L}^p(\mu) := \{f : \Omega \to \mathbb{R},\ \mathcal{F}-\text{measurable},\ ||f||_p < +\infty\}, \quad ||f||_p := \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$$

$L^p(\mu)$ is the quotient of $\mathcal{L}^p(\mu)$ by the equivalent relation "equalitity μ a.e in Ω".

(a) Let $\{f_n\}_{n\in\mathbb{N}}$ a Cauchy sequence in $L^p(\mu)$. Show that it exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that $||f_{n_{k+1}} - f_{n_k}||_p \leq 2^{-k}$.

(b) Let

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \quad \text{and} \quad g = \sum_{i=0}^{+\infty} |f_{n_{i+1}} - f_{n_i}|$$

where g has value in $\mathbb{R} \cup \{\infty\}$. Show that: $\forall k \geq 1, ||g_k||_p < 1$. Show that $||g||_p < 1$.

(c) conclude that the sequence

$$f_{n_k} = f_{n_1} + \sum_{i=1}^{k} \left(f_{n_{i+1}} - f_{n_i}\right)$$

is absolute convergent a.s $x \in \Omega$. We call $f(x)$ when it exists and set $f(x) = 0$ otherway. Show that $f_{n_k}(x) \to f(x)$ almost everywhere.

(d) Show that $f(x) \in L^p(\mu)$. Using Fatou lemma, show that $f_n(x) \to f(x)$ in $L^p(\mu)$. Conclude that $L^p(\mu)$ is complete.

Exercise 2.2 Forard price of a stock option
Claudio works in an hedge fund, he wants to be able to buy one Google stock unit in one year at a price that is fixed today. He calls Corrine, who's working in a bank and he asks her. They agree that Corrine will sell a Google stock to Claudio in one year, at a price fixed today. At which price should they agree? Corrine takes a risk : what if the Google stock double in one year ? How can she hedge herself against this risk ? How can she be sure that she will be able to give him a Google stock in return of this price fixed in advance ?

Corinne hedge herself by having two "assets", some cash and some Google stocks. Her hedging Portfolio V is composed by a certain amount α of cash C and certain amount β of Google stock S.

$$\underbrace{V}_{\text{portfolio}} = \underbrace{\alpha\ C}_{\text{cash}} + \underbrace{\beta\ S}_{\text{Google stocks}}$$

There is no exchange of money today. She doesn't have any money. She can borrow at the bank. If she borrows 1\$ from the bank she have to pay back e^r\$ in one year. The price of the Google stock price today is S_0 (known) and in one year S_1. The hedging portfolio has two states, one today V_0 and one in one year V_1.

$$V_0 = \alpha_0\ C_0 + \beta_0\ S_0$$
$$V_1 = \alpha_1\ C_1 + \beta_1\ S_1$$

(a) She doesn't have any money today, so the value of her hedging portfolio is zero. Find a **self financing strategy** that permits Corrine to fill her duties. Fill the following table . In one year she should give to Claudio a Google stock, hence the S_1.

	today	in one year
cash	*	*
Google stocks	*	S_1
portfolio value	0	

(b) What should be the value of the portfolio in one year ? Why ? conclude the price K at which Corrine accepts to sell a Stock option to Claudio. This is called the forward price.

The (long) **forward contract** is an agreement at time 0 which allows an agent (Claudio) to buy (from Corrine) an asset (the Google Stock) at time 1 at a predetermined *delivery price* K. A forward contract on asset S has the payoff $f(S_1) = S_1 - K$ at time 1. No asset is exchanged at time 0.

Exercise 2.3 Call-Put Parity

Let $(\tilde{S}_0, \tilde{S}_1)$ be a *one step binomial model.* At time 0 the price of the stock is $\tilde{S}_0$. At time 1, there is two posssibilities, the stock price goes up and $\tilde{S}_1 = S_0 u$, or goes down and $\tilde{S}_1 = S_0 d$. The discounted price $S_1 = e^{-r}\tilde{S}_1$ and $S_0 = \tilde{S}_0$

Let have a payoff function $f : \mathbb{R}^+ \to \mathbb{R}^+$. In financial terms, $f(S) = (S-K)^+$ is the payoff function of a (long) *European call option with strike* K and $f(S) = (K-S)^+$ is the payoff function of a (long) *European put option with strike* K.

(a) Construct a self-financing strategy $\varphi \mathrel{\hat{=}} (V_0, \vartheta)$ such that $V_1(\varphi) = e^{-r} f(\tilde{S}_1) \quad P$-a.s..

Hint: Reduce to solving two linear equations.

(b) Prove the *put-call parity*:

$$V_0^c - V_0^p = S_0 - e^{-r} K$$

Give an economic interpretation of this equation.

(c) Compute the limits $\lim_{K\to\infty} V_0$, $\lim_{K\to 0} V_0$, in the case of call and put options.

Exercise 2.4 Python

(a) **Black Sholes for a put option.** Write the function that compute the Black Scholes closed formula of a put option. The payoff of a put option is $f(S_N) = (K - S_N)_+$.

(b) **Monte Carlo.** The price of an European derivative at time 0 is

$$price(N, S_0, K, r, \sigma) = E[e^{-rN} f(S_N)]$$

We have seen, that there is a closed formula for computing the call and put option. There is also another method, called *Monte Carlo.* As $(S_N^i)_i$ are iid, from the law of large numbers [1] we have that

$$e^{-rN} \sum_{i=0}^{M} \frac{f(S_N^i)}{M} \underset{M\to\infty}{\to} E[e^{-rN} f(S_N)]$$

The idea is to simulate a large number of stock prices S_N^i, then calculate the payoff in each scenario (ie for each S_N^i compute $f(S_N^i)$), and finally find the average over all $f(S_N^i)$. Follow the instructions on the file monte carlo and compute the price of a call and a put option using the Monte Carlo method. The number of simulations M is called paths number.

[1] X_i iid then $\frac{X_1 + \cdots + X_M}{M} \to E[X_1]$

```python
from math import exp, sqrt
import random


def monte_carlo_price(maturity, spot, strike, rate, vol, paths_number,
    payoff_fct):
  ...
  return exp(-rate * maturity) * price
```