Introduction to Mathematical Finance

Solution sheet 2

Solution 2.1

(a) $\{f_n\}_{n\in\mathbb{N}}$ a Cauchy sequence in $L^p(\mu)$: $\exists n_1 : \forall n, m \geq n_1, ||f_n - f_m||_p \leq 2^{-1}$.Then $\exists n_2 \geq n_1 : \forall n, m \geq n_2, ||f_n - f_m||_p \leq 2^{-2}$ and so on $\exists n_k \geq n_{k-1} : \forall n, m \geq n_k, ||f_n - f_m||_p \leq 2^{-k}$.

(b) $||g_k||_p = ||\sum_{i=1}^k |f_{n_{i+1}} - f_{n_i}|\ ||_p \leq \sum_{i=1}^k ||f_{n_{i+1}} - f_{n_i}||_p \leq \sum_{i=1}^k 2^{-i} < 1$.
g_k is a increasing positive sequence, such that $\sup_k ||g_k||_p < \infty$ and $g_k \to g$. From the monotone convergence theorem: $g_k \to g$ in L^p and so $||g||_p = lim_{k\to\infty}||g_k||_p < 1$.

(c) As $||g||_p < 1$, $||g||_p < \infty$. So f_{n_k} is absolute convergent and $f_{n_k}(x) \to f(x)$.

(d) $|f| < |f_{n_1}| + |g|$ so $||f||_p < ||\ |f_{n_1}| + |g|\ ||_p < ||f_1||_p + ||g||_p < \infty$. So $f(x) \in L^p$.
$||f - f_n||_p^p = \int |f - f_n|^p d\mu = \int \liminf_{i\to\infty} |f_{n_i} - f_n|^p \mu \overset{*}{\leq} \liminf_{i\to\infty} \int |f_{n_i} - f_n|_p^p \mu \leq \liminf_{i\to\infty} ||f_{n_i} - f_n||_p^p$. As $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, $||f_{n_i} - f_n||_p^p \to 0$, ie $f_n \to f$ in $L^p(\mu)$.
For all $\{f_n\}_{n\in\mathbb{N}}$ Cauchy sequences in $L^p(\mu)$, the limite f is in $L^p(\mu)$, so $L^p(\mu)$ is complete.

Solution 2.2

(a) The strategy of Corrine should be the following. She should buy a Google stock today at the price S_0\$, and for that she must borrow S_0\$. In one year, she should rembourse the bank of S_0\$ plus the interest rates, ie $S_0\ e^r$\$.

	today	in one year
cash	S_0	$-S_0\ e^r$
Google stocks	$-S_0$	S_1
portfolio value	0	**0**

This can be written in fowlling way:

$$V_0 = (S_0) \times 1 + (-1) \times S_0$$
$$V_1 = (-S_0) \times e^r + (1) \times S_1$$

So the stratedy of today : $(\alpha_0, \beta_0) = (-S_0, 1)$ and in one year :$(\alpha_1, \beta_1) = (S_0, 1)$.

(b) From the **no arbitrage principle**, $V_1 = 0$. That means that $S_1 = e^r S_0$. This is the forward price of the Google stock.

Solution 2.3

(a) A self-financing strategy $\varphi \mathrel{\hat{=}} (V_0, \vartheta)$ satisfies

$$V_1(\varphi) = e^{-r} f(\tilde{S}_1) \quad P\text{-a.s.} \tag{1}$$

if and only if we have

$$V_0 + \vartheta_1 \Delta S_1 = e^{-r} f(\tilde{S}_1) \quad P\text{-a.s.}\,.$$

Since S only takes two values, $\Delta S_1 = S_1 - S_0 = e^{-r} S_0 x - S_0 = S_0 e^{-r}(x - e^r)$, for $x = \{u, d\}$. Hence we obtain the following linear equation.

$$V_0 + \vartheta_1 S_0 e^{-r}(u - e^r) = e^{-r} f(S_0 u) \tag{2}$$
$$V_0 + \vartheta_1 S_0 e^{-r}(d - e^r) = e^{-r} f(S_0 d)\,. \tag{3}$$

Subtracting the two equations, multiplying by $(1 + r)$ and dividing by S_0 yields

$$\begin{aligned} & \vartheta_1 S_0(u - d) = f(S_0 u) - f(S_0 d) \\ \Leftrightarrow \quad & \vartheta_1 = \frac{f(S_0 u) - f(S_0 d)}{S_0 u - S_0 d}\,. \end{aligned} \tag{4}$$

Plugging this into (2) yields after rearranging

$$\begin{aligned} V_0 &= -\vartheta_1 S_0 e^{-r}(u - e^r) + e^{-r} f(S_0 u) \\ &= e^{-r}\left(-\vartheta_1 S_0(u - e^r) + f(S_0 u)\right) \\ &= e^{-r}\left(-\frac{f(S_0 u) - f(S_0 d)}{S_0 u - S_0 d} S_0(u - e^r) + f(S_0 u)\right) && (5) \\ &= e^{-r}\left(-(f(S_0 u) - f(S_0 d))\frac{u - e^r}{u - d} + f(S_0 u)\right) && (6) \\ &= e^{-r}\left(\frac{e^r - d}{u - d} f(S_0 u) + \frac{u - e^r}{u - d} f(S_0 d)\right)\,. && (7) \end{aligned}$$

(b) For fixed $K > 0$, by distinguishing the two cases $S \geq K$ and $S < K$, we easily get

$$(S - K)^+ - (K - S)^+ = S - K$$

Thus, using the formulas of V_0 for a call and a put option, we get :

$$\begin{aligned} V_0^c - V_0^p &= e^{-r}\left(\frac{e^r - d}{u - d}(S_0 u - K) + \frac{u - e^r}{u - d}(S_0 d - K)\right) \\ &= e^{-r}\left(S_0\left(\frac{e^r - d}{u - d} u + \frac{u - e^r}{u - d} d\right) - K\left(\frac{e^r - d}{u - d} + \frac{u - e^r}{u - d}\right)\right) \\ &= e^{-r}\left(S_0 e^r - K\right) = S_0 - e^r K && (8) \end{aligned}$$

The economic interpretation is that buying a stock and a put option with strike K and maturity T is equivalent to buying a call option with the same strike and the same maturity and a zero-coupon bond with the same maturity and face value K.

(c) For fixed $S \geq 0$ we have

$$\begin{aligned} \lim_{K\to\infty} f^c(S) &= \lim_{K\to\infty}(S - K)^+ = 0 \\ \lim_{K\to 0} f^p(S) &= \lim_{K\to 0}(K - S)^+ = S \end{aligned}$$

Therefore we have

$$\begin{aligned} \lim_{K\to\infty} V_0^c &= \lim_{K\to\infty} e^{-r}\left(\frac{e^r - d}{u - d} f^c(S_0 u) + \frac{u - e^r}{u - d} f^c(S_0 d)\right) \\ &= e^{-r}\left(\frac{e^r - d}{u - d}\lim_{K\to\infty} f^c(S_0 u) + \frac{u - e^r}{u - d}\lim_{K\to\infty} f^c(S_0 d)\right) = 0 \\ \lim_{K\to 0} V_0^p &= e^{-r}\left(\frac{e^r - d}{u - d}\lim_{K\to 0} f^p(S_0 u) + \frac{u - e^r}{u - d}\lim_{K\to 0} f^p(S_0 d)\right) = S && (9) \end{aligned}$$

Using the Call Put parity (8) $V_0^c - V_0^p = S_0 - e^{-r}K$, we have

$$\begin{aligned} \lim_{K\to\infty} V_0^p &= \lim_{K\to\infty} \left(e^{-r}K - S_0 + V_0^c\right) = +\infty, \\ \lim_{K\to 0} V_0^c &= \lim_{K\to 0} \left(S_0 - e^{-r}K + V_0^p\right) = 0\,. \end{aligned} \tag{10}$$

Solution 2.4

```python
from math import exp, sqrt
import random


def monte_carlo_price(maturity, spot, strike, rate, vol, paths_number,
    payoff_fct):
  price = 0
  for i in range(paths_number):
    price += payoff_fct(spot * exp((rate - vol ** 2 / 2) * maturity + vol *
    sqrt(maturity) * random.gauss(0,1)), strike)
  price /= paths_number
  return exp(-rate * maturity) * price
```