ETH Zürich
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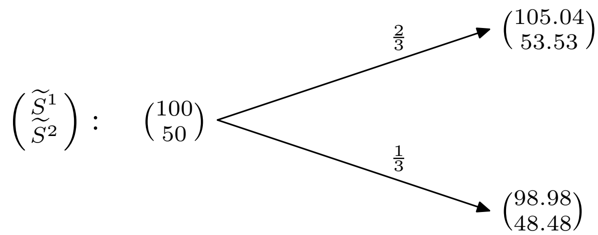
Introduction to Mathematical Finance

Exercise sheet 3

Exercise 3.1 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ consisting of a bank account and two stocks. The stock price movements of $\widetilde{S}^1$ and $\widetilde{S}^2$ are described by the following tree, where the numbers beside the branches denote transition probabilities. Assume that the risk-free rate is given by $r = 0.01$.



(a) Show that the financial sub-markets $(\widetilde{S}^0, \widetilde{S}^1)$ and $(\widetilde{S}^0, \widetilde{S}^2)$ are free of arbitrage by constructing the equivalent martingale measures Q^1 for S^1 and Q^2 for S^2.

Hint: Write down the tree for the discounted stock prices S^1 and S^2.

(b) Show that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is not free of arbitrage by explicitly constructing an *arbitrage opportunity*.

Hint: Calculate the expectation of S_1^2 under the equivalent martingale measure Q^1 for S^1.

(c) By which number do you have to replace 105.04 in the stock price movement of $\widetilde{S}^1$ so that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is free of arbitrage?

Exercise 3.2 Binomial market

(a) Let $(\tilde{S}_0, \tilde{S}_1)$ be a *one step binomial model.* At time 0 the price of the stock is $\tilde{S}_0$. At time 1, there is two possibilities, the stock price goes up and $\tilde{S}_1 = S_0 u$, or goes down and $\tilde{S}_1 = S_0 d$. The discounted price $S_1 = e^{-r}\tilde{S}_1$ and $S_0 = \tilde{S}_0$. Let have a payoff function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Construct a self-financing strategy ξ such that $V_1(\xi) = e^{-r} f(\tilde{S}_1) \quad P$-a.s..

(b) This binomial model can be extended to any arbitrary number of periods n, the *multi-step binomial model* by setting $S_n^0 = e^{rn}$. At each time, there is two possibilities, the stock price goes up and $\tilde{S}_{k+1} = \tilde{S}_k u$, or goes down and $\tilde{S}_{k+1} = \tilde{S}_k d$. Let f be a payoff of the form $f(\tilde{S}_n^1)$. The arbitrage free price of such a payoff at time k can then be written as $v(k, \tilde{S}_k^1)$.
Show that the function $v(k, \tilde{S}_k^1)$ fulfills the following backward recursion formula:

$$v(k, x) = e^{-r}\left(qv(k+1, xu) + (1-q)v(k+1, xd)\right), \tag{1}$$

where

$$q := \frac{e^r - d}{u - d}$$

and

$$v(n, x) = f(x).$$

(c) Write a *pseudo-code* of a multi-steps binomial tree that computes the arbitrage free price of an derivative with payoff f.

Exercise 3.3 Call non-decreasing with respect to maturity
Consider a general multi-period market and denote by $C(t, K)$ the payoff $(S_t^1 - K)^+$ at time t. Call t the maturity of the option. Assume that these options are traded, that $r \geq 0$, and that the market is free of arbitrage.

Fix K and show that the price of such call options is non-decreasing as a function of maturity.

Exercise 3.4 Python - Call and Put Using the Black and Scholes closed formula plot the price of a call and put option with respect the spot price. Follow the instruction on the plot.py file. See the first example of a call option.

```
from math import sin, cos

import closed_formula
import monte_carlo
from payoff import call, put


def plot_example():
  x_values = np.linspace(-5, 5, 100)
  sin_values = [sin(x) for x in x_values]
  cos_values = [cos(x) for x in x_values]
  plt.plot(x_values, sin_values, label='sin')
  plt.plot(x_values, cos_values, label='cos')
  plt.xlabel('x')
  plt.ylabel('f(x)')
  plt.legend()
  plt.show()
```

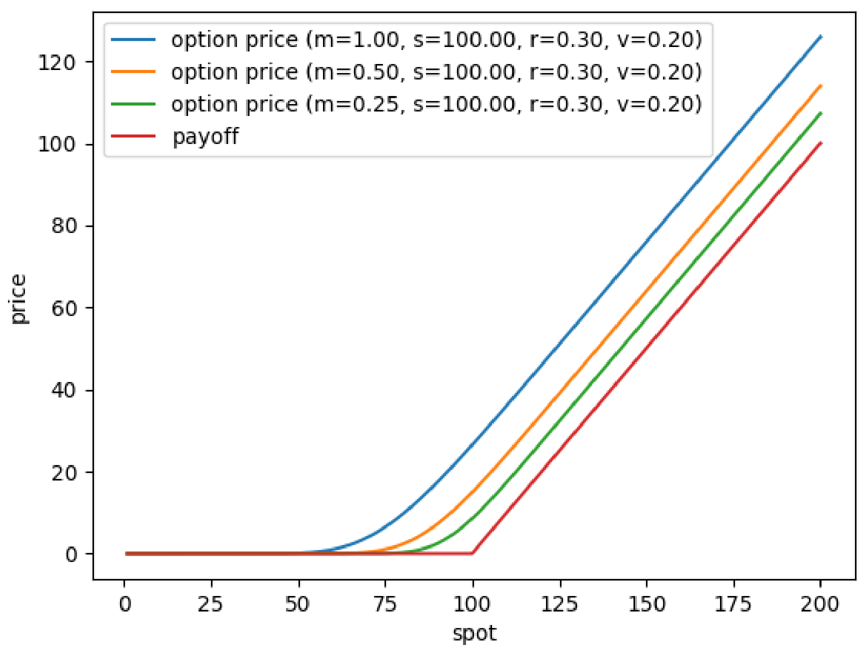


Figure 1: **Call option price** with strike 100, rate 30%, volatility 20%. For different maturities : 1Y (year), 6M (months) and 3M.