IMF Serie 3

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Exercise 3.1 Consider a financial market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ consisting of a bank account and two stocks. The stock price movements of $\widetilde{S}^1$ and $\widetilde{S}^2$ are described by the following tree, where the numbers beside the branches denote transition probabilities.

$$\begin{pmatrix}\widetilde{S}^1\\ \widetilde{S}^2\end{pmatrix}: \quad \begin{pmatrix}100\\ 50\end{pmatrix} \begin{cases} \xrightarrow{\frac{2}{3}} \begin{pmatrix}105.04\\ 53.53\end{pmatrix} \\[2ex] \xrightarrow{\frac{1}{3}} \begin{pmatrix}98.98\\ 48.48\end{pmatrix} \end{cases}$$

~~[illegible]~~ the risk-free rate is given by $r = 0.01$.

(b) Show that the financial sub-markets $(\widetilde{S}^0, \widetilde{S}^1)$ and $(\widetilde{S}^0, \widetilde{S}^2)$ are free of arbitrage by constructing the equivalent martingale measures Q^1 for S^1 and Q^2 for S^2.

Hint: Write down the tree for the discounted stock prices S^1 and S^2.

(c) Show that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is not free of arbitrage by explicitly constructing an *arbitrage opportunity.*

Hint: Calculate the expectation of S_1^2 under the equivalent martingale measure Q^1 for S^1.

(d) By which number do you have to replace 105.04 in the stock price movement of $\widetilde{S}^1$ so that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is free of arbitrage?

b) Proof: $S_t^i = \frac{\widetilde{S}_t^i}{\widetilde{S}_t^0}$ $\rightsquigarrow$ we take Q^i with probs. q_1^i that "up" and q_2^i that "down" $\rightsquigarrow$ we have $q_1^i + q_2^i = 1$ and since $Q^i \sim P$ we also have $q_j^i > 0$

$\rightsquigarrow$ S^i has to be martingale under Q^i $\Leftrightarrow$ $S_0^i \overset{!}{=} \mathbb{E}_{Q^i}(S_1^i \mid \mathcal{F}_0) = \mathbb{E}_{Q^i}(S_1^i) = S_1^i(\omega_1) \cdot q_1^i + S_1^i(\omega_2)\, q_2^i$

$\rightsquigarrow$ inserting $q_2^i = 1 - q_1^i$ $\Rightarrow$ $S_1^i(\omega_1)\, q_1^i + S_1^i(\omega_2) - S_1^i(\omega_2)\, q_1^i = S_0^i$ $\Leftrightarrow$ $q_1^i = \frac{S_0^i - S_1^i(\omega_2)}{S_1^i(\omega_1) - S_1^i(\omega_2)}$ $\rightsquigarrow$ $q_2^i = 1 - q_1^i = \frac{S_1^i(\omega_1) - S_0^i}{S_1^i(\omega_1) - S_1^i(\omega_2)}$

$\rightsquigarrow$ computing the values we get: $S_1^1 = \begin{cases}104\\ 98\end{cases}$, $S_1^2 = \begin{cases}53\\ 48\end{cases}$, $S_1^0 = 1$ $\rightsquigarrow$ $q_1^1 = \frac{100-98}{104-98} = \frac{2}{6} = \frac{1}{3}$, $q_2^1 = \frac{2}{3}$, $q_1^2 = \frac{50-48}{53-48} = \frac{2}{5}$, $q_2^2 = \frac{3}{5}$

Thm 1.55

$\Rightarrow$ for both sub-markets exist EMM $\Rightarrow$ both sub-markets are arbitrage free $\blacksquare$

c) Proof: $\varrho_1^0 = -50$, $\varrho_1 = 1$, $\varrho_1^2 = -1$ $\Rightarrow$ $\widetilde{V}_0 = -50 + 100 - 50 = 0$

$\rightsquigarrow$ $\widetilde{V}_1(\omega_2) = -50 \cdot (1.01) + 98.98 - 48.48 = 0$ $\rightsquigarrow$ $\widetilde{V}_1(\omega_1) = -50 \cdot (1.01) + 105.04 - 53.53 = 1.01$

$\Rightarrow$ the portfolio ϱ is an arbitrage portfolio

2nd possibility:

$\rightsquigarrow$ $E_{Q^1}(S_1^2) = 53 \cdot \frac{1}{3} + 48 \cdot \frac{2}{3} = \frac{53 + 96}{3} = \frac{149}{3} < 50$ $\Rightarrow$ we should sell S^2

$\rightsquigarrow$ same portfolio as above yields: $V_1(\omega_1) = -50 + 104 - 53 = 1$, $V_1(\omega_2) = -50 + 98 - 48 = 0$ $\Rightarrow$ arbitrage

d) sol: we see that $E_{Q^2}(S_1^1) = \hat{S}_1^1(\omega_1) \cdot \frac{2}{5} + 98 \cdot \frac{3}{5} \overset{!}{=} 100$ (condition that S_1^1 is also martingale under Q^2 and therefore Q^2 is a EMM for the whole market $\Rightarrow$ (NA)) yields $\underline{\hat{S}_1^1(\omega_1)} = \frac{5}{2}\left(100 - 98 \cdot \frac{3}{5}\right) = 250 - 147 = \underline{103}$

$\Rightarrow$ $\underline{\hat{\widetilde{S}}_1^1(\omega_1) = \hat{S}_1^1(\omega_1) \cdot (1+r) = 104.03}$

Exercise 3.2 Binomial market

(a) Let $(\tilde{S}_0, \tilde{S}_1)$ be a *one step binomial model.* At time 0 the price of the stock is $\tilde{S}_0$. At time 1, there is two posssibilities, the stock price goes up and $\tilde{S}_1 = S_0 u$, or goes down and $\tilde{S}_1 = S_0 d$. The discounted price $S_1 = e^{-r}\tilde{S}_1$ and $S_0 = \tilde{S}_0$. Let have a payoff function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Construct a self-financing strategy $\varphi \mathrel{\hat{=}} (V_0, \vartheta)$ such that $V_1(\varphi) = e^{-r} f(\tilde{S}_1)$ P-a.s..

(b) This binomial model can be exteded to any arbitrary number of periods n, the *multi-step binomial model* by setting $\widetilde{S}_n^0 = e^{rn}$. At each time, there is two posssibilities, the stock price goes up and $\tilde{S}_{k+1} = \tilde{S}_k u$, or goes down and $\tilde{S}_{k+1} = \tilde{S}_k d$. Let f be a payoff of the form $f(\widetilde{S}_n^1)$. The arbitrage free price of such a payoff at time k can then be written as $\widetilde{V}_k^f = v(k, \widetilde{S}_k^1)$. show that the function $v(k, \widetilde{S}_k^1)$ fulfills the following backward recursion formula:

$$v(k,x) = e^{-r}\left(q v(k+1, xu) + (1-q) v(k+1, xd)\right), \tag{1}$$

where

$$q := \frac{e^r - d}{u - d}$$

and

$$v(n,x) = f(x).$$

(c) Write a *pseudo-code* of a multi-steps binomial tree that computes the arbitrage free price of an derivative with payoff f.

a) sol: $V_1(\varphi) = V_0 + \varrho_1^1 \cdot (S_1 - S_0) = V_0 + \varrho_1^1 \begin{cases} e^{-r} S_0 \cdot u - S_0 \\ e^{-r} S_0 \cdot d - S_0 \end{cases}$

$\rightsquigarrow$ $e^{-r} f(\widetilde{S}_1) = e^{-r}\begin{cases} f(S_0 u) \\ f(S_0 \cdot d) \end{cases}$

$\rightsquigarrow$ $V_1(\varphi) \overset{!}{=} e^{-r} f(\widetilde{S}_1)$ $\Leftrightarrow$ $e^{-r}\begin{cases} f(S_0 \cdot u) \\ f(S_0 \cdot d) \end{cases} = V_0 + \varrho_1^1 \cdot S_0 \begin{cases} e^{-r} u - 1 \\ e^{-r} d - 1 \end{cases}$

$\Leftrightarrow$ $V_0 = e^{-r}\begin{cases} f(S_0 u) \\ f(S_0 d) \end{cases} - \varrho_1^1 \cdot S_0 \begin{cases} e^{-r} u - 1 \\ e^{-r} d - 1 \end{cases}$

$\rightsquigarrow$ $e^{-r} f(S_0 u) - \varrho_1^1 S_0 (e^{-r} u - 1) = V_0 = e^{-r} f(S_0 \cdot d) - \varrho_1^1 S_0 (e^{-r} d - 1)$

$\Leftrightarrow$ $\underline{\varrho_1^1} = \frac{e^{-r}(f(S_0 \cdot d) - f(S_0 \cdot u))}{S_0 (e^{-r} d - e^{-r} \cdot u)} = \underline{\frac{f(S_0 u) - f(S_0 d)}{S_0 (u - d)}}$

$\rightsquigarrow$ $\underline{V_0} = e^{-r} f(S_0 \cdot u) - \frac{f(S_0 \cdot u) - f(S_0 \cdot d)}{\cancel{S_0} (u - d)} \cdot \cancel{S_0} (e^{-r} \cdot u - 1) =$

$= \frac{e^{-r} (u - d) f(S_0 \cdot u)}{u - d} - \frac{(f(S_0 \cdot u) - f(S_0 \cdot d))\,(e^{-r} u - 1)}{u - d} =$

$= \underline{\frac{f(S_0 \cdot d)(e^{-r} u - 1) + f(S_0 \cdot u) \cdot (1 - e^{-r} d)}{u - d}}$

$\rightsquigarrow$ ϱ is self-financing if $V_0 = \varrho_1^0 \cdot 1 + \varrho_1^1 \cdot S_0$ $\Leftrightarrow$ $\underline{\varrho_1^0 = V_0 - \varrho_1^1 S_0}$

b) Proof: $\widetilde{V}_k^f = v(k, \widetilde{S}_k)$, $\widetilde{S}_{k+1} = \begin{cases} \widetilde{S}_k \cdot u \\ S_k \cdot d \end{cases}$ $\rightsquigarrow$ $V_{k+1} = V_k + \varrho \cdot (S_{k+1} - S_k)$ $\Leftrightarrow$ $e^{-r(k+1)} \widetilde{V}_{k+1} = e^{-rk} \widetilde{V}_k + \varrho\left(e^{-r(k+1)} \widetilde{S}_{k+1} - e^{-rk} \widetilde{S}_k\right)$

$\Leftrightarrow$ $e^{-r} \widetilde{V}_{k+1} = \widetilde{V}_k + \varrho \cdot (e^{-r} \widetilde{S}_{k+1} - \widetilde{S}_k) = v(k, \widetilde{S}_k) + \varrho \cdot \begin{cases} e^{-r} \widetilde{S}_k u - \widetilde{S}_k \\ e^{-r} \widetilde{S}_k \cdot d - \widetilde{S}_k \end{cases} = v(k, \widetilde{S}_k) + \varrho \cdot \widetilde{S}_k \begin{cases} e^{-r} u - 1 \\ e^{-r} d - 1 \end{cases}$ (1)

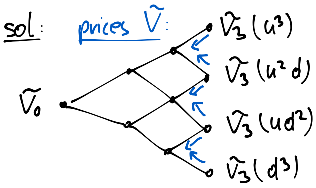
$\rightsquigarrow$ $e^{-r} \widetilde{V}_{k+1} = e^{-r} v(k+1, \widetilde{S}_{k+1}) = e^{-r}\begin{cases} v(k+1, \widetilde{S}_k \cdot u) \\ v(k+1, \widetilde{S}_k \cdot d) \end{cases}$ (2)

⇝ combining (1) & (2) we get: $e^{-r}\begin{Bmatrix} v(k+1, \tilde{S}_k \cdot u) \\ v(k+1, \tilde{S}_k \cdot d) \end{Bmatrix} = v(k, \tilde{S}_k) + g \cdot \tilde{S}_k \cdot \begin{bmatrix} e^{-r}u - 1 \\ e^{-r}d - 1 \end{bmatrix}$

⇝ this is exactly the same equation-system as in a) $\Rightarrow v(k, \tilde{S}_k) = \dfrac{v(k+1, \tilde{S}_k \cdot u)(1 - e^{-r}d) + v(k+1, \tilde{S}_k \cdot d) \cdot (e^{-r}u - 1)}{u - d} =$

$= e^{-r}\left[v(k+1, \tilde{S}_k \cdot u) \cdot q + v(k+1, \tilde{S}_k \cdot d)(1-q)\right]$, for $q = \dfrac{e^r - d}{u - d}$, ⇝ $v(n, \tilde{S}_n) = \tilde{V}_n = f(\tilde{S}_n)$ ∎

c) <u>sol</u>: <u>prices $\tilde{V}$</u>:



⇝ for $k = n$: $\tilde{V}_n(\cdot)$ is known, since we know $f(\tilde{S}_n)$, i.e.: $\tilde{V}_n(u^j d^{n-j}) = f(\tilde{S}_n(u^j d^{n-j}))$

⇝ $u^j d^{k-j}$ corresponds to those $\omega \in \Omega$ (in general more than only 1) for which the path goes up j times and down $(k-j)$ times (up to time k)

<u>Alg</u>: <u>input</u>: vector F having stored the values of $f(\tilde{S}_n)$ s.t. $F_0 = f(\tilde{S}_n(u^n d^0)), \dots, F_j = f(\tilde{S}_n(u^{n-j} d^j)), \dots, F_n = f(\tilde{S}_n(u^0 d^n))$
rate r, u, d

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V = zeros(1, n+1),  q = (e^r - d)/(u - d)
V = F
for i = 1 to n
    for j = 0 to (n-i)
        V_j = e^{-r} [V_j · q + V_{j+1} · (1-q)]
    end
end
return V_0
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Exercise 3.3 Call non-decreasing with respect to maturity
Consider a general multi-period market and denote by $C(t, K)$ the payoff $(S_t^1 - K)^+$ at time t. Call t the maturity of the option. Assume that these options are traded, that $r \geq 0$, and that the market is free of arbitrage.

Fix K and show that the price of such call options is non-decreasing as a function of maturity.

<u>Proof</u>: ⇝ since the market is arbitrage free, i.e. (NA) holds, we know by FTAP (Thm 5.16) that an EMM Q exists.

⇝ if $(S_t^1 - K)^+$ is attainable then we know by Thm 5.33 that it has a unique price, which is $\mathbb{E}_Q((S_t^1 - K)^+)$ (see Thm 5.30) otherwise, if it is not attainable, the price is not unique, but given by $\mathbb{E}_{\tilde{Q}}((S_t^1 - K)^+)$, $\forall \tilde{Q}$ EMM

⇝ in either way we want to show that $\forall$ $s \leq t \in \mathbb{N}$ and $\forall \tilde{Q}$ EMM: $\mathbb{E}_{\tilde{Q}}((S_s^1 - K)^+) \leq \mathbb{E}_{\tilde{Q}}((S_t^1 - K)^+)$

<u>Claim 1</u>: $((S_n - K)^+)_{n \in \mathbb{N}}$ is a submartingale under $\tilde{Q}$ <u>Proof 1</u>: $(S_n)_n$ is martingale $\Rightarrow (S_n - K)_n$ is marting. ⇝ $g: x \mapsto x^+ = \max\{x, 0\}$ is a convex function $\Rightarrow (g(S_n - K))_n = ((S_n - K)^+)_n$ is a submartingale (see Prop Th., Prop 3.13) □ (Claim 1)

⇝ therefore we have: $\mathbb{E}_{\tilde{Q}}((S_t^1 - K)^+) \overset{t \geq s}{=} \mathbb{E}_{\tilde{Q}}\big(\underbrace{\mathbb{E}_{\tilde{Q}}((S_t^1 - K)^+ \mid \mathcal{F}_s)}_{\geq (S_s^1 - K)^+, \text{ by submartingale property}}\big) \geq \mathbb{E}_{\tilde{Q}}((S_s^1 - K)^+)$ $\forall t \geq s \in \mathbb{N}$ ∎