ETH Zürich
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Prof. Josef Teichmann

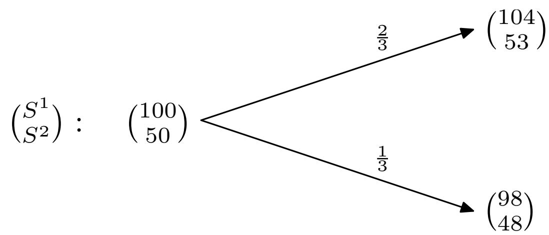
Coordinator
Calypso Herrera

Introduction to Mathematical Finance

Solution sheet 3

Solution 3.1

(a) The tree for the discounted stock prices S^1 and S^2 is given by



Any probability measure Q equivalent to P on $\mathcal{F}_1$ can be described by a probability vector $(q_1, q_2) \in (0,1)^2$, where $q_1 = Q[S_1 = 98]$ and $q_2 = Q[S_1 = 104]$. Since $\mathcal{F}_0$ is trivial, S^1 is a Q^1-martingale and S^2 is a Q^2-martingale if and only if $E_{Q^1}\left[S_1^1\right] = S_0^1$ and $E_{Q^2}\left[S_1^2\right] = S_0^2$, respectively. The latter is equivalent to

$$\begin{aligned} & 98q_1^1 + 104(1 - q_1^1) = 100 && \text{and} && 48q_1^2 + 53(1 - q_1^2) = 50 \\ \Leftrightarrow \quad & 6q_1^1 = 4 && \text{and} && 5q_1^2 = 3 \\ \Leftrightarrow \quad & q_1^1 = 2/3 && \text{and} && q_1^2 = 3/5. \end{aligned}$$

In conclusion, Q^1 and Q^2 are respectively described by the probability vectors $(2/3, 1/3)$ and $(3/5, 2/5)$, respectively. Note that Q^1 and Q^2 are the unique equivalent martingale measures for S^1 and S^2, respectively.

(b) We have

$$E_{Q^1}\left[S_1^2\right] = 2/3 \times 48 + 1/3 \times 53 \approx 49.67 < 50 = S_0^2\,. \tag{1}$$

If the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ were free of arbitrage, by the *Fundamental Theorem of Asset Pricing* (FTAP) there would be an equivalent martingale measure Q, which a fortiori would be an equivalent martingale measure for the sub-market $(\widetilde{S}^0, \widetilde{S}^1)$, too. Given that $(\widetilde{S}^0, \widetilde{S}^1)$ has the unique equivalent martingale measure Q^1, this would imply that $Q = Q^1$. Therefore, since $E_{Q^1}\left[S_1^2\right] < S_0^2$, we may on the one hand conclude that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is not free of arbitrage, and on the other hand that $\widetilde{S}^2$ is "overpriced". Hence, we should short-sell, say, one share of $\widetilde{S}^2$ and invest the money into $\widetilde{S}^1$ and the bank account. Therefore, we take the strategy $\hat{\xi}_1 = (\xi_1^1, \xi_1^2) = (c, -1)$, where $c > 0$ is still to be determined. Then we have [1]

$$(\xi \bullet S)_1 = \begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} \cdot \begin{pmatrix} \Delta S_1^1 \\ \Delta S_1^2 \end{pmatrix} = \begin{cases} c \times (98 - 100) - 1 \times (48 - 50) = 2 - 2c & \text{on } \{S_1 = 98\} \\ c \times (104 - 100) - 1 \times (53 - 50) = 4c - 3 & \text{on } \{S_1 = 104\}\,, \end{cases} \tag{2}$$

[1] cumulative gains and losses $(\xi \bullet S)_t = \sum_{k=1}^{T} \bar{\xi}_k \cdot (\bar{S}_k - \bar{S}_{k-1}) = \sum_{k=1}^{t} \bar{\xi}_k \cdot \Delta\bar{S}_k$

If the initial value of the portfolio is $V_0(\xi) = 0$ then ξ is an arbitrage opportunity if and only if $V_1(\xi) \geq 0 \Leftrightarrow (\xi \bullet S)_1 \geq 0$ [2]

$$\begin{aligned} \Leftrightarrow \quad & 2 - 2c \geq 0 \quad \text{and} \quad 4c - 3 \geq 0 \\ \Leftrightarrow \quad & c \leq 1 \quad \text{and} \quad c \geq 3/4\,, \end{aligned} \tag{3}$$

where at least one of the two inequalities is strict. Thus, ξ is an arbitrage opportunity if and only if $c \in [0.75, 1]$.

(c) If we replace 105.04 in the stock price movement of $\widetilde{S}^1$ by s, then the argument in part (b) shows that the market $(\widetilde{S}^0, \widetilde{S}^1, \widetilde{S}^2)$ is free of arbitrage if and only if Q^2, the equivalent martingale measure for the sub-market $(\widetilde{S}^0, \widetilde{S}^2)$, is an equivalent martingale measure for the whole market, i.e. if and only if S^1 is a Q^2-martingale. Since $\mathcal{F}_0$ is trivial, the latter is equivalent to

$$\begin{aligned} E_{Q^2}\left[S_1^1\right] = S_0^1 \quad & \Leftrightarrow \quad 3/5 \times 98 + 2/5 \times \frac{s}{1+r} = 100 \\ & \Leftrightarrow \quad s = 103 \times 1.01 = 104.03\,. \end{aligned} \tag{4}$$

Solution 3.2 Binomial market

(a) A self-financing strategy ξ satisfies

$$V_1(\xi) = e^{-r} f(\tilde{S}_1) \quad P\text{-a.s.} \tag{5}$$

if and only if we have

$$V_0 + \xi_1 \Delta S_1 = e^{-r} f(\tilde{S}_1) \quad P\text{-a.s.}\,.$$

Since S only takes two values, $\Delta S_1 = S_1 - S_0 = e^{-r} S_0 x - S_0 = S_0 e^{-r}(x - e^r)$, for $x = \{u, d\}$. Hence we obtain the following linear equation.

$$V_0 + \xi_1 S_0 e^{-r}(u - e^r) = e^{-r} f(S_0 u) \tag{6}$$

$$V_0 + \xi_1 S_0 e^{-r}(d - e^r) = e^{-r} f(S_0 d)\,. \tag{7}$$

Subtracting the two equations, multiplying by e^r and dividing by S_0 yields

$$\begin{aligned} \xi_1 S_0 (u - d) &= f(S_0 u) - f(S_0 d) \\ \Leftrightarrow \quad \xi_1 &= \frac{f(S_0 u) - f(S_0 d)}{S_0 u - S_0 d}\,. \end{aligned} \tag{8}$$

Plugging this into (6) yields after rearranging

$$\begin{aligned} V_0 &= -\xi_1 S_0 e^{-r}(u - e^r) + e^{-r} f(S_0 u) \\ &= e^{-r}\left(-\xi_1 S_0 (u - e^r) + f(S_0 u)\right) \\ &= e^{-r}\left(-\frac{f(S_0 u) - f(S_0 d)}{S_0 u - S_0 d} S_0 (u - e^r) + f(S_0 u)\right) \\ &= e^{-r}\left(-(f(S_0 u) - f(S_0 d))\frac{u - e^r}{u - d} + f(S_0 u)\right) \\ &= e^{-r}\left(\frac{e^r - d}{u - d} f(S_0 u) + \frac{u - e^r}{u - d} f(S_0 d)\right)\,. \end{aligned}$$

[2] $V_1(\xi) = V_0(\xi) + (\xi \bullet S)_1$

(b) At maturity, the price of the option is equal to its payoff: $v(n, x) = f(x)$. At the previous date, we apply (a).

$$\begin{aligned} v(n-1, \tilde{S}_{n-1}) &= e^{-r}\left(\frac{e^r - d}{u-d} f(\tilde{S}_{n-1}u) + \frac{u - e^r}{u-d} f(\tilde{S}_{n-1}d)\right) \\ &= e^{-r}\left(qv(n, \tilde{S}_{n-1}u) + (1-q)v(n, \tilde{S}_{n-1}d)\right) . \end{aligned}$$

At a previous date $k-1$, knowing $v(k, xu)$ and $v(k, xd)$ and by applying (a), we get

$$v(k-1, x) = e^{-r}\left(qv(k, xu) + (1-q)v(k, xd)\right) .$$

(c)

```
function PRICE(T, S_0, K, r, σ, n, payoff)
    δt ← T/n
    u ← e^{σ√δt}
    d ← 1/u
    q_u ← (e^{rδt} − d)/(u − d)                ▷ set the risk neutral probability

    for k in [0 .. n-1] do                     ▷ forward: set the stock prices
        for i in [0 .. k] do
            S(k, i) ← S_0 u^{k−i} d^i
                                               ▷ backward: compute the option prices
    for i in [0 .. n] do                       ▷ first initialize values at maturity
        v(n − 1, i) ← PAYOFF(S(n − 1, i), K)

    for k in [n-2.. 0] do                      ▷ then move to earlier steps
        for i in [0 .. k] do
            v(k, i) ← e^{−rδt}(q_u v(k + 1, i) + (1 − q_u)v(k + 1, i + 1))
    return v(0, 0)
```

Solution 3.3 Call non-decreasing with respect to maturity

Assume the opposite, i.e., that there exist two time points $t_1 < t_2$ such that $\pi(C(t_1, K)) > \pi(C(t_2, K))$ and construct an arbitrage opportunity. We will only need to trade at time 0 and time t_1. Without loss of generality, we may assume that the prices below are discounted.

Time 0 Sell $C(t_1, K)$, buy $C(t_2, K)$, and deposit the difference ξ_1^0 in the bank account. This comes with initial cost 0.

Time t_1 At time t_1 we have two cases:

A: $S^1_{t_1} \le K$. Do not do anything.

B: $S^1_{t_1} > K$. Take a short position in S^1 and deposit K in the bank account.

Time t_2 If we are in scenario A it does not matter what happens at t_2 since we will get a non-negative payoff and have no short positions. We are therefore left with B.

C: $S^1_{t_2} < K$. Close the short position in S^1 at the cost $S^1_{t_2}$, which is less than what we deposited. More precisely, the value

$$V_2 = \xi_1^0 - S^1_{t_2} + K > \xi_1^0 > 0.$$

D: $S^1_{t_2} \ge K$. Get $S^1_{t_2} - K$ from the call option. Close the position in S^1 and the bank account. The value in this scenario is therefore ξ_1^0.

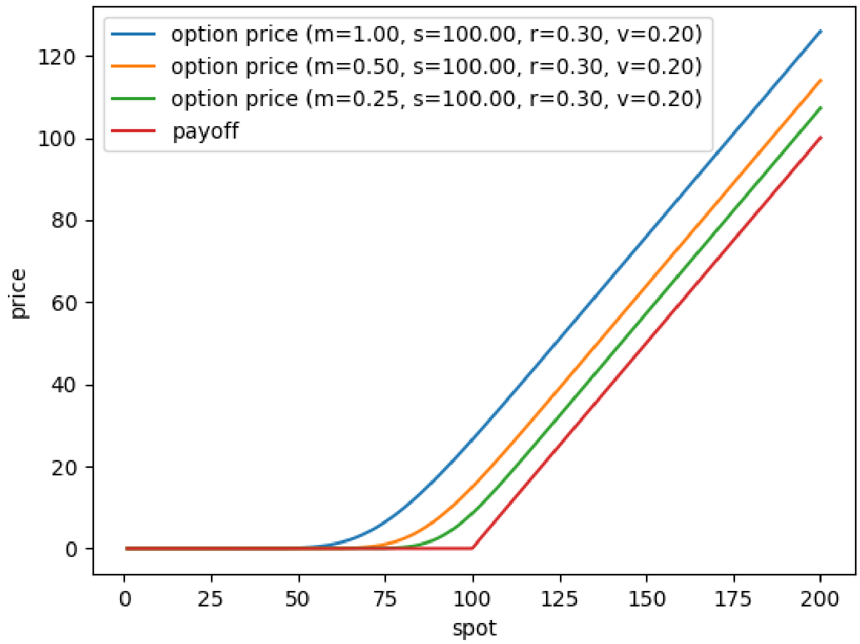


Figure 1: **Call option price** with strike 100, rate 30%, volatility 20%. For different maturities : 1Y (year), 6M (months) and 3M.

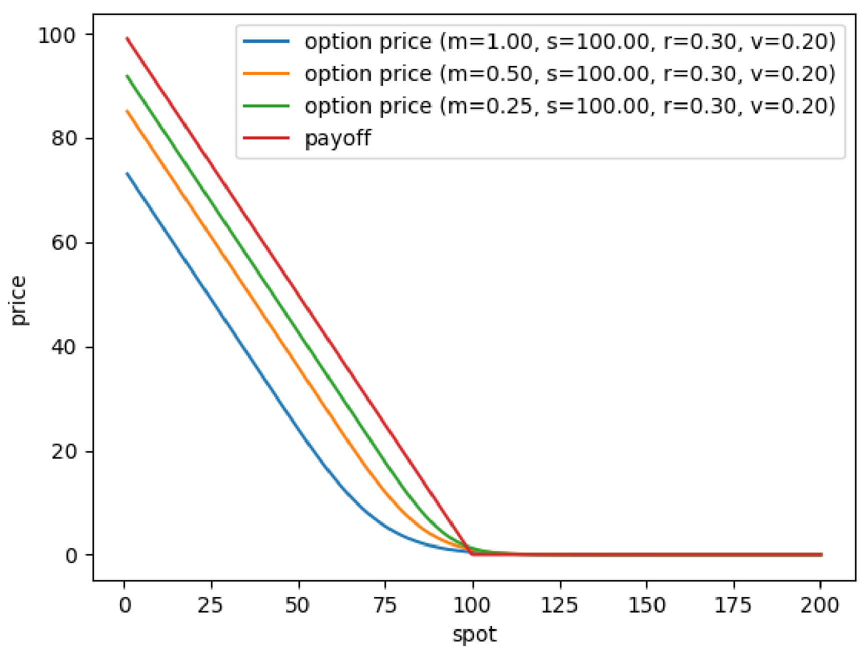


Figure 2: **Put option price** with strike 100, rate 30%, volatility 20%. For different maturities : 1Y, 6M and 3M.

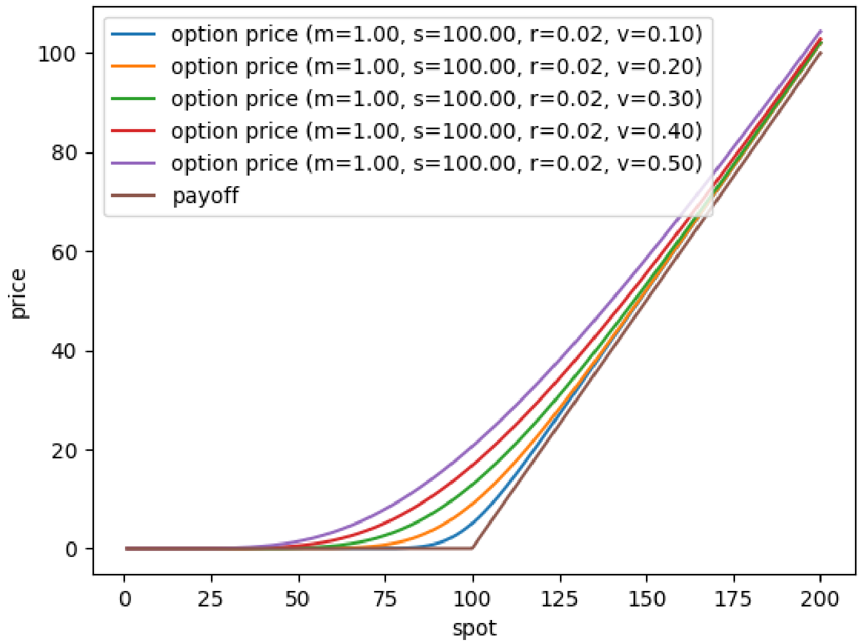


Figure 3: Call option price with maturity 1Y, strike 100, rate 2%. For different **volatilities** : 10, 20, 30 and 50%.

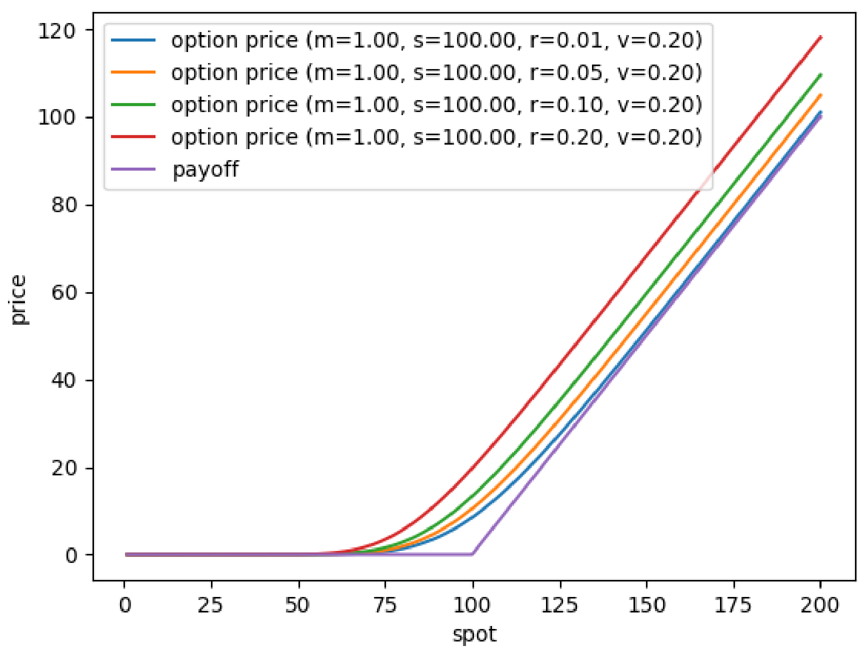


Figure 4: Call option price with maturity 1Y, strike 100, rate 2% and volatility 0.2. For different **rate** : 1, 5, 10 and 20%.

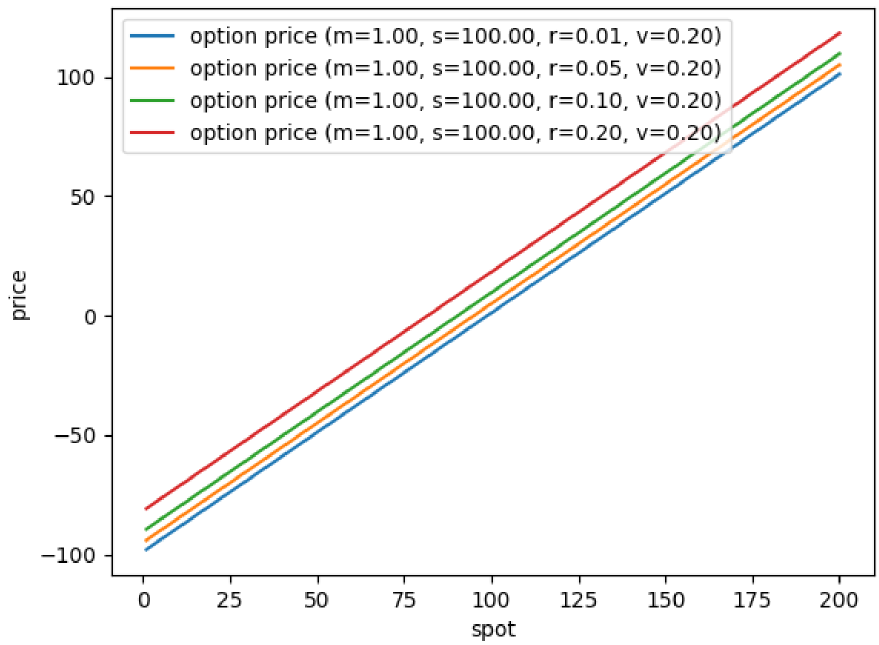


Figure 5: **Call - Put price** with maturity 1Y, strike 100, rate 2% and volatility 0.2. For different **rate** : 1, 5, 10 and 20%.

We have thus constructed a portfolio which has initial value $V_0 = 0$ and final value $V_{t_2} \geq 0$ P-a.s.—an arbitrage opportunity. By the assumption of no arbitrage, this is a contradiction.

Solution 3.4 Python - Call and Put