Coordinator
Calypso Herrera

Introduction to Mathematical Finance

Exercise sheet 4

Exercise 4.1 Let $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T})$ be a filtered probability space and $X = (X_k)_{k=0,\dots,T}$ a *discounted* price process. Show that the following are equivalent:

(a) S satisfies (NA).

(b) For each $k = 0, \dots, T-1$, the one-period market (X_k, X_{k+1}) on $(\Omega, \mathcal{F}_{k+1}, P, (\mathcal{F}_k, \mathcal{F}_{k+1}))$ satisfies (NA). Give an economic interpretation of this result.

Hint: Prove the contraposition of the direction "(b) ⇒ (a)". Argue via induction on T.

Exercise 4.2 Consider a multi-step *binomial market*. Let have First fix $r > -1$ and let $S^0_k = (1+r)^k$. Now define $S^1_0 = 1$ and

$$S^1_k = \prod_{i=1}^{k} R^1_i, \quad k = 1, \dots, T,$$

where the R^1_k are i.i.d. and

$$P[R^1_k = 1+u] = 1 - P[R^1_k = 1+d] \in (0,1),$$

for $u > d$.

Suppose now, for the sake of the exercise, that $r = 0$, $u = 0.5$ as well as $d = -0.5$ and consider the strategy (V_0, ξ) given by

$$\xi_k = \frac{1}{S^1_{k-1}} 2^k 1_{\{k \le \tau\}},$$

where $\tau = \inf\{k | R^1_k = 1+u\} \wedge T$.

(a) Calculate the biggest loss over all time points to see how it depends on T. Conclude that the strategy would not be admissible if $T = \infty$.

(b) Suppose that $T = \infty$ and calculate the value of the strategy at the stopping time τ.

Remark: The mathematical term 'martingale' has one of its origins in this type of strategy, also called martingales.

Exercise 4.3 Consider a market with trading dates $k = 0, \dots, T$, with N traded assets on the probability space $(\Omega, \mathcal{F}, P)$ and the filtration given by $\mathbb{F} = (\mathcal{F}_k)_{k=0,\dots,T}$, i.e., a general multiperiod market.

For any strategy ξ, we define the process $\widetilde{C} = (\widetilde{C}_k)_{k=0,\dots,T}$ by

$$\widetilde{C}_k(\xi) := \widetilde{V}_k(\xi) - (\xi \bullet S)_k.$$

(a) Show that [1]

$$\Delta \widetilde{C}_{k+1}(\xi) = \Delta \xi_{k+1} \cdot S_k,$$

for $k = 1, \dots, T-1$.

[1] $\Delta A_{k+1} := A_{k+1} - A_k$

(b) Show that ξ is self-financing if and only if

$$\widetilde{C}_k(\xi) = \widetilde{C}_0(\xi),$$

for $k = 1, \ldots, T$.

Hint: Be careful with definitions at the first time point.

Remark: The process $\widetilde{C}$ is called the *cost process* for ξ.

Exercise 4.4 Python - Binomial tree

Compute the price of an option using a binomial tree.

```
def binomial_price(maturity, spot, strike, rate, vol, steps_number, payoff_fct, graph_name=None):
  """Compute the binomial price. Draw graph if graph_name is given.
  """
  deltaT = maturity / steps_number
  discount_factor = exp(-rate * deltaT)
  up = exp(vol * sqrt(deltaT))
  down = 1 / up
  proba_up = (1/discount_factor - down) / (up - down)
  proba_down = 1 - proba_up
  steps = range(steps_number)
  # See create_graph docstring for an explanation on how
  # spot_prices and option_prices are structured.
  spot_prices = [[None] * (i+1) for i in steps]
  option_prices = [[None] * (i+1) for i in steps]

  # Forward: set the stock prices
  for n in steps:
    for i in range(n+1):
      spot_prices[n][i] = 0.  # TODO

  # Backward: compute the option prices
  # First initialize values at maturity ([-1] means last element)
  for i, spot_price in enumerate(spot_prices[-1]):
    option_prices[-1][i] = 0.  # TODO

  # Then move to earlier steps
  for n in reversed(steps[:-1]):  # t[:-1] is to remove last element
    for i in range(n+1):
      option_prices[n][i] = 0.  # TODO

  if graph_name:
    create_graph(graph_name, spot_prices, option_prices)

  return option_prices[0][0]
```

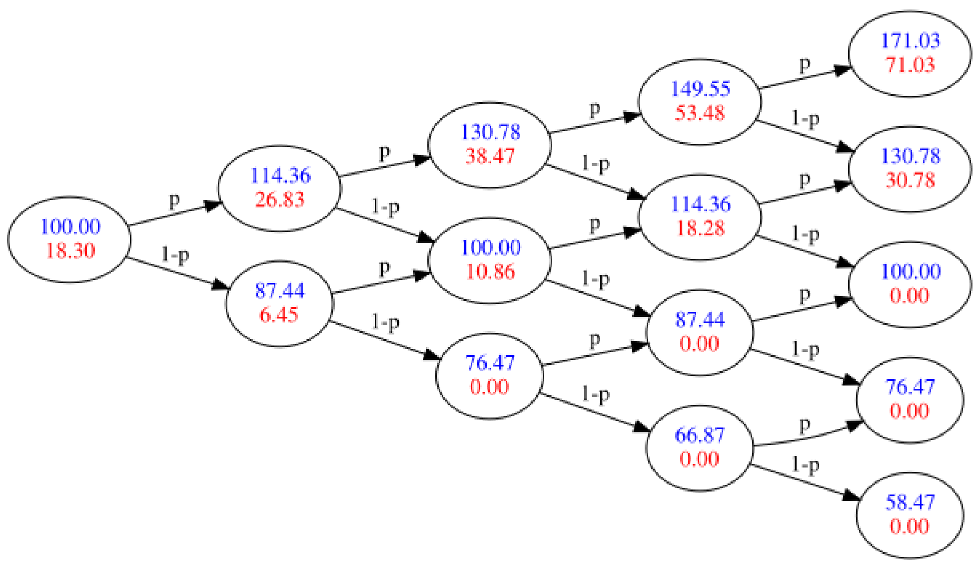

Figure 1: **Binomial tree for a Call option** with strike 100, rate 20%, volatility 30%.