ETH Zürich
D-Math, Spring 2017
Prof. Josef Teichmann

Coordinator
Calypso Herrera

Introduction to Mathematical Finance

Solution sheet 4

Solution 4.1 "(a) $\Rightarrow$ (b)": We prove the contraposition. Let ξ_k be an $\mathcal{F}_{k-1}$-measurable random variable such that $\xi_k \cdot \Delta X_k \geq 0$ P-a.s. and $P[\xi_k \cdot \Delta X_k > 0] > 0$. Extending ξ_k to a predictable process $\widehat{\xi}$ via

$$\widehat{\xi}_j := \begin{cases} \xi_k, & \text{if } j = k, \\ 0, & \text{for } j \in \{1, \dots, T\} \backslash \{k\}, \end{cases}$$

we obtain that $(\xi \bullet X)_T \geq 0$ P-a.s. and hence also $P[(\xi \bullet X)_T > 0] > 0$. This means that arbitrage in the "small" market yields arbitrage in the "big" market.

"(b) $\Rightarrow$ (a)": Again, we prove the contraposition. Let ξ be an arbitrage opportunity, i.e.

$$(\xi \bullet X)_T \geq 0 \quad P\text{-a.s.} \quad \text{and} \quad P[(\xi \bullet X)_T > 0] > 0\,.$$

We claim that there exist $k \in \{1, \dots, T\}$ and $A \in \mathcal{F}_{k-1}$ such that $P[A] > 0$, $\mathbb{1}_A \xi_k\ \Delta X_k \geq 0$ P-a.s. and $P[\mathbb{1}_A \xi_k \cdot \Delta X_k > 0] > 0$.

Proof: We prove the statement by induction on T. For $T = 1$, the situation is trivially satisfied. Suppose the assertion holds for $T - 1$. We distinguish three possibilities:

1. $P[(\xi \bullet X)_{T-1} < 0] > 0$,
2. $P[(\xi \bullet X)_{T-1} = 0] = 1$ and
3. $(\xi \bullet X)_{T-1} \geq 0$ P-a.s. and $P[(\xi \bullet X)_{T-1} > 0] > 0$.

In the case of $P[(\xi \bullet X)_{T-1} < 0] > 0$, we define $A := \{(\xi \bullet X)_{T-1} < 0\}$ and the strategy

$$\xi_k^A := \begin{cases} \mathbb{1}_A \xi_T, & k = T, \\ 0, & \text{for } k \in \{1, \dots, T-1\}\,. \end{cases}$$

Because $(\xi^A \bullet X)_T = \mathbb{1}_A \left((\xi \bullet X)_T - (\xi \bullet X)_{T-1}\right)$, this strategy obviously implies $(\xi^A \bullet X)_T \geq 0$ P-a.s. and $P\left[(\xi^A \bullet X)_T > 0\right] > 0$.
In the case of $P[(\xi \bullet X)_{T-1} = 0] = 1$, the choice $A = \Omega$ obviously yields an arbitrage opportunity in the one-period market (X_{T-1}, X_T) on $(\Omega, \mathcal{F}_T, P, (\mathcal{F}_{T-1}, \mathcal{F}_T))$.
In the remaining case $(\xi \bullet X)_{T-1} \geq 0$ P-a.s. and $P[(\xi \bullet X)_{T-1} > 0] > 0$, we apply the inductive hypothesis.

It remains to give the interpretation. The result tells us that in order for a financial market to be free of arbitrage, it is necessary and sufficient that the local models (X_k, X_{k+1}) are arbitrage-free. Thus, the notion of (NA), which is a priori globally defined, turns out to be of local nature. If, on the other hand, one knows the Fundamental Theorem of Asset Pricing, then one realizes that this local notion translates into nothing else then the following:
In order to check whether a, say adapted and integrable, process X is a martingale, it suffices to check whether

$$E_Q[X_{k+1} \,|\, \mathcal{F}_k] = X_k \quad \forall k \text{ (local behaviour) instead of } E_Q[X_T \,|\, \mathcal{F}_k] = X_k \quad \forall k \text{ (global behaviour).}$$

Solution 4.2

(a) If all price changes have been negative until $k-1$, then an additional depreciation causes the change

$$\begin{aligned}\xi_k \Delta S_k^1 &= \xi_k (S_k^1 - S_{k-1}^1)\\ &= \xi_k \left(-\frac{1}{2} S_{k-1}^1\right)\\ &= \frac{1}{S_{k-1}^1} 2^k \left(-\frac{1}{2} S_{k-1}^1\right)\\ &= -2^{k-1}\end{aligned}$$

of the gains process. Since the biggest loss is attained if all price changes are negative, it is given by the sum of these changes until T, i.e., by

$$(\xi \bullet S)_T = \sum_{k=1}^{T} -2^{k-1} = -(2^T - 1).$$

Hence, for any $a \in \mathbb{R}$, there exists a time horizon such that (V_0, ξ) is not a-admissible.

(b) At τ we know that all but the last price change have been losses. Thus,

$$(\xi \bullet S)_\tau = (\xi \bullet S)_{\tau-1} + \frac{1}{S_{\tau-1}^1} 2^\tau \frac{S_\tau^1}{2} = 2^{\tau-1} - (2^{\tau-1} - 1) = 1.$$

Solution 4.3

(a) We are done if we can show that $\Delta \widetilde{V}_{k+1}(\xi) - \Delta(\xi \bullet S)_{k+1} = \Delta \xi_{k+1} \cdot S_k$ for $k = 1, \ldots, T-1$. By the definitions,

$$\begin{aligned}\Delta \widetilde{V}_{k+1}(\xi) - \Delta(\xi \bullet S)_{k+1} &= \xi_{k+1} \cdot S_{k+1} - \xi_k \cdot S_k - \xi_{k+1} \cdot \Delta S_{k+1}\\ &= -\xi_k \cdot S_k + \xi_{k+1} \cdot S_k\\ &= \Delta \xi_{k+1} \cdot S_k,\end{aligned}$$

which means we are done.

(b) The property $\widetilde{C}_k(\xi) = \widetilde{C}_0(\xi)$ for $k = 1, \ldots, T$ is equivalent to

$$\Delta \widetilde{C}_{k+1} = 0,$$

for $k = 0, \ldots, T-1$.

In view of (a), this condition is stronger than ξ being self-financing so we need the observation that $\widetilde{C}_1(\xi) = \widetilde{C}_0(\xi)$ always holds. Indeed,

$$\widetilde{C}_1(\xi) = \widetilde{V}_1(\xi) - (\xi \bullet S)_1 = \xi_1 \cdot S_1 - \xi_1 \cdot \Delta S_1 = \xi_1 \cdot S_0 = \widetilde{V}_0(\xi) = \widetilde{C}_0(\xi),$$

i.e., $\Delta \widetilde{C}_1 = 0$ is always true. Combining this observation with (a), the definition of ξ being self-financing is equivalent to $\Delta \widetilde{C}_{k+1} = 0$ for $k = 0, \ldots, T-1$. By the first equivalence, we are done.