Introduction to Mathematical Finance

Exercise sheet 5

Exercise 5.1 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be an *arbitrage-free* financial market with time horizon T and assume that the bond satisfies $\widetilde{S}_k^0 = (1+r)^k$ for $k = 0, \ldots, T$ with $r \geq 0$. Denote the set of all equivalent martingale measures for S^1 by $P_e(S^1)$. Fix $K > 0$ and let $k \in \{1, \ldots, T\}$. The payoff of a *European call option* on $\widetilde{S}^1$ with strike K and maturity k is denoted by C_k^E and given by

$$C_k^E = \left(\widetilde{S}_k^1 - K\right)^+ ,$$

whereas the payoff an *Asian call option* on $\widetilde{S}^1$ with strike K and maturity k is denoted by C_k^A and given by

$$C_k^A := \left(\frac{1}{k} \sum_{j=1}^{k} \widetilde{S}_j^1 - K \right)^+ .$$

(a) Fix $Q \in P_e(S^1)$. Show that the function $\{1, \ldots, T\} \to \mathbb{R}^+$, $k \mapsto E_Q\left[\frac{C_k^E}{\widetilde{S}_k^0}\right]$ is increasing.

Hint: Use *Jensen's inequality* for conditional expectations.

(b) Fix $Q \in P_e(S^1)$. Show that for all $k = 1, \ldots, T$ we have

$$E_Q\left[\frac{C_k^A}{\widetilde{S}_k^0}\right] \leq \frac{1}{k} \sum_{j=1}^{k} E_Q\left[\frac{C_j^E}{\widetilde{S}_j^0}\right] .$$

(c) Fix $Q \in P_e(S^1)$. Deduce that for all $k = 1, \ldots, T$ we have

$$E_Q\left[\frac{C_k^A}{\widetilde{S}_k^0}\right] \leq E_Q\left[\frac{C_k^E}{\widetilde{S}_k^0}\right] .$$

Exercise 5.2 In this exercise we consider the probability space $(\Omega, \mathcal{F}, P)$ and a market given by

$$\pi^0 = 1, \qquad \pi^1 = 1,$$
$$S^0 = e^r, \qquad S^1 = e^Y,$$

where Y follows a standard normal distribution under P. If you want, you may assume that $\Omega = \mathbb{R}$. Define

$$\Pi^b(C^{\text{call}}) := \left\{ E^b\left[\frac{C^{\text{call}}}{e^r}\right] : S^1 \text{ is binomially distributed under } P^b, E^b\left[\frac{S^1}{e^r}\right] = \pi^1 \right\}.$$

This is the set of arbitrage free prices under some measure for which S^1 is binomially distributed.

The goal of this exercise is to show that

$$\Pi^b(C^{\text{call}}) \subseteq \left[\Pi_{\inf}(C^{\text{call}}), \Pi_{\sup}(C^{\text{call}})\right] .$$

(a) Construct a sequence of martingale measures absolutely continuous to P that converges weakly to a martingale measure under which S^1 is binomially distributed.

(b) Construct convex combinations of this sequence and P^* to show the inclusion above.

Exercise 5.3

(a) Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be a function. We define $f^* : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that

$$f^*(y) := \sup_{x\in\mathbb{R}} \{yx - f(x)\} \qquad \forall y \in \mathbb{R}\,.$$

(1) Show that f^* is a closed convex function.[1]

(2) Show that if f is a closed convex function then $f^{**} = f$.[2]

(b) Let now $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and $f^* : X^* \to \mathbb{R} \cup \{\pm\infty\}$[3] defined as follow

$$f^*(x^*) := \sup_{x\in X} \{< x^*, x > -f(x)\} \qquad \forall x^* \in X^*\,.$$

Show that if f is a closed convex function then $f^{**} = f$.

(c) Let $C \subset X$ be a non-empty closed convex cone. The polar $C^\circ = \{x^* \in X^* |\ < x^*, x > \leq 0 \quad \forall x \in C\}$ of C is then another non-empty closed convex cone and $(C^\circ)^\circ = C$. Show that if

$$f(x) = \delta(x|C) := \begin{cases} 0 & x \in C \\ \infty & \text{else} \end{cases}$$

is the *indicator function* of a cone C then $(C^\circ)^\circ = C \iff f^{**} = f$.

Exercise 5.4 Python - Path dependent derivative

(a) Change the binomial price function to handle a double knock-out barrier call option with payoff $f(S_T) = (S_T - K)_+ 1_{a\leq S_t\leq b, 0\leq t\leq T}$. Same for a double knockout barrier put option.

(b) Draw the graph for a cal and a put, for steps number = 10.

(c) Compare with Monte Carlo (see compare.py).

[1] A *convex set* is a subset C of a vector space X if for all $x, y \in C, \lambda \in [0, 1]$ we have that $\lambda x + (1-\lambda)y \in C$. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be a function, we define the *epigraph* as $\text{epi}(f) = \{(x, \alpha) : x, \alpha \in \mathbb{R} \text{ and } f(x) \leq \alpha\}$. f is said to be a *convex function* if epi(f) is a convex set. f is said to be a *closed function* if epi(f) is a closed set.

[2] If f is a lower semi-continuous and convex, then $f(x) := \sup_{a\leq f}\{a(x)\}$ where the supremium is taken over all continuous affine functionals on X.

[3] The dual space of X denoted X^* is the space of continuous linear functionals on X.