Introduction to Mathematical Finance

Solution sheet 5

Solution 5.1

(a) It clearly suffices to show that for all $k = 1, \ldots, T-1$ we have

$$E_Q\left[\frac{C^E_{k+1}}{\widetilde{S}^0_{k+1}}\right] \geq E_Q\left[\frac{C^E_k}{\widetilde{S}^0_k}\right]. \tag{1}$$

Fix $k \in \{1, \ldots, T-1\}$. Using the *tower property* of conditional expectations, *Jensen's inequality* for conditional expectations (for the convex function $x \mapsto x^+$), the fact that S^1 is a Q-martingale and $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{C^E_{k+1}}{\widetilde{S}^0_{k+1}}\right] &= E_Q\left[\left(S^1_{k+1} - \frac{K}{(1+r)^{k+1}}\right)^+\right] \\
&= E_Q\left[E_Q\left[\left(S^1_{k+1} - \frac{K}{(1+r)^{k+1}}\right)^+ \middle| \mathcal{F}_k\right]\right] \\
&\geq E_Q\left[\left(E_Q\left[S^1_{k+1} - \frac{K}{(1+r)^{k+1}} \middle| \mathcal{F}_k\right]\right)^+\right] \\
&= E_Q\left[\left(S^1_k - \frac{K}{(1+r)^{k+1}}\right)^+\right] \\
&\geq E_Q\left[\left(S^1_k - \frac{K}{(1+r)^k}\right)^+\right] \\
&= E_Q\left[\frac{C^E_k}{\widetilde{S}^0_k}\right].
\end{aligned} \tag{2}$$

(b) Since the function $x \mapsto x^+$ is convex, we have for $k = 1, \ldots, T$

$$\begin{aligned}
C^A_k &= \left(\frac{1}{k}\sum_{j=1}^k \widetilde{S}^1_j - K\right)^+ = \left(\sum_{j=1}^k \frac{1}{k}\left(\widetilde{S}^1_j - K\right)\right)^+ \\
&\leq \sum_{j=1}^k \frac{1}{k}\left(\widetilde{S}^1_j - K\right)^+ = \frac{1}{k}\sum_{j=1}^k C^E_j.
\end{aligned} \tag{3}$$

By *linearity* and *monotonicity* of conditional expectations and since $r \geq 0$, we get

$$\begin{aligned}
E_Q\left[\frac{C^A_k}{\widetilde{S}^0_k}\right] &= E_Q\left[\frac{C^A_k}{(1+r)^k}\right] \leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^k}\right] \\
&\leq \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{(1+r)^j}\right] = \frac{1}{k}\sum_{j=1}^k E_Q\left[\frac{C^E_j}{\widetilde{S}^0_j}\right].
\end{aligned} \tag{4}$$

(c) Putting the results of (a) and (b) together yields for $k = 1, \ldots, T$

$$E_Q\left[\frac{C_k^A}{\widetilde{S}_k^0}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{C_j^E}{\widetilde{S}_j^0}\right] \leq \frac{1}{k}\sum_{j=1}^{k} E_Q\left[\frac{C_k^E}{\widetilde{S}_k^0}\right] = E_Q\left[\frac{C_k^E}{\widetilde{S}_k^0}\right]. \tag{5}$$

Solution 5.2 We will consider measures P^ε fulfilling

$$d(P^\varepsilon \circ (S^1)^{-1}) = f^\varepsilon d\mu,$$

for some Borel function f^ε and where μ denotes the Lebesgue measure—measures under which S^1 has the probability density function f^ε.

Assume for now that P^ε can be constructed from $P^\varepsilon \circ (S^1)^{-1}$ and proceed to construct f^ε appropriately. Set

$$f^\varepsilon(s) = C_d^\varepsilon 1_{[d,d+\varepsilon)}(s) + C_u^\varepsilon 1_{[u,u+\varepsilon)}(s)$$

for some u and d such that $u > e^r > d$. We will require u and d to be in the image of $S^1 : \Omega \to \mathbb{R}$. Then the martingale conditions take the form

$$\begin{aligned}
1 &= \int_{\mathbb{R}} f^\varepsilon(s)ds = C_d^\varepsilon \varepsilon + C_u^\varepsilon \varepsilon,\\
1 &= \int_{\mathbb{R}} e^{-r} s f^\varepsilon(s)ds = C_d^\varepsilon e^{-r}\frac{\varepsilon^2 + 2\varepsilon d}{2} + C_u^\varepsilon e^{-r}\frac{\varepsilon^2 + 2\varepsilon u}{2}.
\end{aligned}$$

This is a linear system of equations in C_d^ε and C_u^ε which is solved by

$$\begin{bmatrix} C_d^\varepsilon \\ C_u^\varepsilon \end{bmatrix} = \frac{1}{e^{-r}\varepsilon^2(u-d)} \begin{bmatrix} e^{-r}\frac{\varepsilon^2+2\varepsilon u}{2} & -\varepsilon \\ -e^{-r}\frac{\varepsilon^2+2\varepsilon d}{2} & \varepsilon \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\varepsilon(u-d)} \begin{bmatrix} \frac{\varepsilon}{2} + u - e^r \\ e^r - d - \frac{\varepsilon}{2} \end{bmatrix}$$

Note that by the constraints on u and d, $f^\varepsilon > 0$ is indeed a probability density function.

Under this sequence of measures P^ε, the law of S^1 converges weakly to

$$\frac{u - e^r}{u-d}\delta_d + \frac{e^r - d}{u-d}\delta_u.$$

Call the corresponding probability measure P^b. This measure makes S^1 a martingale.

Now let

$$P_{\text{conv}}^\varepsilon = \varepsilon P^* + (1-\varepsilon)P^\varepsilon$$

for $1 \gg \varepsilon > 0$. Then, since $P_{\text{conv}}^\varepsilon$ is a convex combination of martingale measures, $P_{\text{conv}}^\varepsilon$ is itself a martingale measure. Furthermore, since P^ε is absolutely continuous to P and $P^* \approx P$, $P_{\text{conv}}^\varepsilon \approx P$ and it is an equivalent martingale measure. It follows that

$$\int_\Omega \frac{C^{\text{call}}}{e^r} dP_{\text{conv}}^\varepsilon \in \left(\Pi_{\inf}(C^{\text{call}}), \Pi_{\sup}(C^{\text{call}})\right).$$

Furthermore,

$$\int_\Omega \frac{C^{\text{call}}}{e^r} dP_{\text{conv}}^\varepsilon = \varepsilon \underbrace{\int_\Omega \frac{C^{\text{call}}}{e^r} dP^*}_{<\infty} + (1-\varepsilon)\int_\Omega \frac{C^{\text{call}}}{e^r} dP^\varepsilon.$$

As P^* does not change with ε, the first term tends to 0, and since all P^ε have their support inside $[0, u+1]$ and C^{call} is bounded on bounded domains, the weak convergence implies that

$$\int_\Omega \frac{C^{\text{call}}}{e^r} dP_{\text{conv}}^\varepsilon \longrightarrow E^b\left[\frac{C^{\text{call}}}{e^r}\right] \in \left[\Pi_{\inf}(C^{\text{call}}), \Pi_{\sup}(C^{\text{call}})\right].$$

Since the choice of u and d was arbitrary, we are done as soon as we have covered our assumption above:

Let us now show we can find P^ε from the law constructed above. With the assumption that $\Omega = \mathbb{R}$ this is straight forward by an appropriate choice of Y. In the general case we see that it will be true whenever $S^1 : \Omega \to \mathbb{R}$ is injective. However, we could construct the measure on the space where we identify two points in Ω if they are mapped to the same value by S^1. Since $\mathcal{F} = \sigma(S^1)$, such a measure directly translates to the original space, which means we are done.
Remark: The conclusion does not depend on the fact that Y has mean 0 and variance 1, but we would have to use another EMM than P^* since it would no longer be one.

Solution 5.3 Fenchel-Moreau Theorem

(a) (1) Define the affine functions $h_x(y) = yx - f(x)$. The affine functions are closed and convex which means that $\mathrm{epi}(h_x)$ is a closed convex set for all $x \in \mathbb{R}$. We write f^* as follow

$$f^*(y) = \sup_{x\in\mathbb{R}}\{yx - f(x)\} = \sup_{x\in\mathbb{R}}\{h_x(y)\}\,.$$

This implies

$$\mathrm{epi}(f^*) = \sup_{x\in\mathbb{R}}\{\mathrm{epi}(h_x)\} = \bigcap_{x\in\mathbb{R}}\{\mathrm{epi}(h_x)\}\,.$$

The intersection of closed convex sets is again closed convex which means that f^* is closed and convex.

Remark: f^* is called the Legendre Transformation.

(2) We use the fact that a closed convex function f can be written as the pointwise supremum of the collection of all affine functions h satisfying $h \le f$.[1]

$$f(x) = \sup\{h(x) = yx - \alpha : h \le f\} \qquad \forall x \in \mathbb{R}$$

This leads to

$$\begin{aligned} yx - \alpha \le f(x) \qquad \forall x \in \mathbb{R} &\iff yx - f(x) \le \alpha \qquad \forall x \in \mathbb{R} \\ &\iff \sup_{x\in\mathbb{R}}\{yx - f(x)\} \le \alpha \\ &\iff f^*(y) \le \alpha \\ &\iff (y,\alpha) \in \mathrm{epi}(f^*)\,. \end{aligned}$$

So we have

$$f(x) = \sup_{(y,\alpha)\in\mathrm{epi}(f^*)}\{yx - \alpha\} \qquad \forall x \in \mathbb{R}\,.$$

For $(y,\alpha) \in \mathrm{epi}(f^*)$ we have that $yx - \alpha \le yx - f^*(x)$, so

$$\begin{aligned} f(x) &= \sup_{y\in\mathbb{R}}\{yx - f^*(x)\} \qquad \forall x \in \mathbb{R} \\ &= f^{**}(x) \qquad \forall x \in \mathbb{R}\,. \end{aligned}$$

(b) By definition of f^*, $f(x) \ge \sup_{x\in X^*}\{<x^*, x> - f^*(x)\}$, $\forall x^* \in X^*$ so $f \ge f^{**}$.
Let a an affine minorant of f, $a \le f$, so $a^* \ge f^*$ and $a^{**} \le f^{**}$. But since a is affine, $a^{**} = a$. So every affine minorant of f is an affine minorant of f^{**}. We know that if f is a l.s.c. and convex, then $f(x) := \sup_{a\le f}\{a(x)\}$ where the supremeium is taken over all continuous affine functionals on X. We conclude that $f \le f^{**}$.

[1] see theorem 12.1 in *Convex Analysis* of R.Tyrrell Rockafellar.

(c) If $f(x) = \delta(x|C)$ for an non empty convex cone then $f^*(x^*) = \delta(x^*|C^\circ)$ for a certain non convex cone which must be closed since f^* is closed. The *conjugate* of $f^*(\cdot) = \delta(\cdot|C^\circ)$ is $f^{**}(\cdot) = \delta(\cdot|C^{\circ\circ})$. So we obtain that

$$\begin{aligned} f(x) = f^{**}(x) \iff & \quad \delta(x|C) = \delta(x|C^{\circ\circ}) \\ \iff & \quad C = C^{\circ\circ}\,. \end{aligned}$$

Remark: Theorem 14.1 in *Convex Analysis* of R.Tyrrell Rockafellar.