⑥

Ex. 6.2 $\rightsquigarrow$ Assumption: $\hat{b} := 1 + \underbrace{b}_{\geq 0} \geq 1 \implies \hat{a} := 1 + \underbrace{a}_{\leq 0} = \frac{1}{\hat{b}} \leq 1$.

(a) Up-and-out call option: $C^{call}_{u\&o} = (S_T - K)_+ \mathbb{1}_{\max_{0\le t\le T} S_t < B}$, where $B > S_0 \vee K$.

- $\pi(C^{call}_{u\&o}) = E^*\left[\frac{C^{call}_{u\&o}}{(1+r)^T}\right] = \frac{1}{(1+r)^T} E^*[C^{call}_{u\&o}]$.
- $E^*[C^{call}_{u\&o}] = E^*[(S_T-K)_+ \mathbb{1}_{\max_{0\le t\le T} S_t < B}] = E^*[(S_T-K)_+] - E^*[(S_T-K)_+ \mathbb{1}_{\max_{0\le t\le T} S_t \ge B}]$

$$\left.\begin{aligned}
&= E^*[(S_T-K)_+] - E^*[(S_T-K)_+ \mathbb{1}_{S_T\ge B}] - E^*[(S_T-K)_+ \mathbb{1}_{\max_{0\le t\le T} S_t \ge B}\mathbb{1}_{S_T<B}]\\
&= E^*[(S_T-K)_+(1-\mathbb{1}_{S_T\ge B})] \qquad - \left(\frac{p^*}{1-p^*}\right)^k\left(\frac{B}{S_0}\right)^2 E^*[(S_T-\tilde{K})_+\mathbb{1}_{S_T<\tilde{B}}]\\
&= E^*[(S_T-K)_+\mathbb{1}_{S_T<B}] \qquad - \text{———— '' ————},\\
&\text{where } \tilde{K} := \left(\frac{S_0}{B}\right)^2 K = \hat{b}^{-2k}K,\quad \tilde{B} := \frac{S_0^2}{B} = S_0\hat{b}^{-k}.
\end{aligned}\right\}\ (i)$$

(i) Script, Example: Up-and-in call option.

$$\Rightarrow \pi(C^{call}_{u\&o}) = \frac{1}{(1+r)^T}\left\{E^*[(S_T-K)_+\mathbb{1}_{S_T<B}] - \left(\frac{p^*}{1-p^*}\right)^k\left(\frac{B}{S_0}\right)^2 E^*[(S_T-\tilde{K})_+\mathbb{1}_{S_T<\tilde{B}}]\right\}.$$ ✓

(b) Down-and-in put option: $C^{put}_{d\&i} = (K-S_T)_+ \mathbb{1}_{\min_{0\le t\le T} S_t \le B}$, where $B < S_0$.

- $\pi(C^{put}_{d\&i}) = E^*\left[\frac{C^{put}_{d\&i}}{(1+r)^T}\right] = \frac{1}{(1+r)^T} E^*[C^{put}_{d\&i}]$.
- $E^*[C^{put}_{d\&i}] = E^*[(K-S_T)_+ \mathbb{1}_{\min_{0\le t\le T} S_T \le B}]$

$$\begin{aligned}
&= E^*[(K-S_T)_+\mathbb{1}_{\min_{0\le t\le T} S_t\le B}\mathbb{1}_{S_T\le B}] + E^*[(K-S_T)_+\mathbb{1}_{\min_{0\le t\le T} S_t\le B}\mathbb{1}_{S_T>B}]\\
&= \underbrace{E^*[(K-S_T)_+\mathbb{1}_{S_T\le B}]}_{\text{can be explicitly computed}} + \underbrace{E^*[(K-S_T)_+\mathbb{1}_{\min_{0\le t\le T} S_t\le B}\mathbb{1}_{S_T>B}]}_{=:(\mathrm{I})}
\end{aligned}$$

- Wlog B lies in the range of possible asset prices with $B < S_0$.

  $\iff \exists k\in\mathbb{N}: B = S_0\hat{b}^{-k} = S_0\hat{a}^k \iff \hat{b}^{2k} = \left(\frac{S_0}{B}\right)^2$

- $-\ S_T > B \iff S_0\hat{b}^{Z_T} > S_0\hat{b}^{-k} \iff Z_T > -k$

  $-\ \min_{0\le t\le T} S_t \le B \iff \min_{0\le t\le T} S_0\hat{b}^{Z_t} \le S_0\hat{b}^{-k} \iff \min_{0\le t\le T} Z_t \le -k \iff \tilde{M}_T \le -k$, where $\tilde{M}_T := \min_{0\le t\le T} Z_t$.

- $(\mathrm{I}) = E^*[(K-S_T)_+\mathbb{1}_{\min_{0\le t\le T} S_t\le B}\mathbb{1}_{S_T>B}] = E^*[(K-S_0\hat{b}^{Z_T})_+\mathbb{1}_{\tilde{M}_T\le -k}\mathbb{1}_{Z_T>-k}]$

  $= \sum_{l\ge1} E^*[(K-S_0\hat{b}^{Z_T})_+\mathbb{1}_{\tilde{M}_T\le -k}\mathbb{1}_{Z_T=-k+l}] = \sum_{l\ge 1}(K-S_0\hat{b}^{-k+l})_+ P^*[\tilde{M}_T\le -k, Z_T=-k+l]$

- Note: We can prove the reflection principle for $P^*[\tilde{M}_t\le -k, Z_T=-k+l]$ analogously to Lem. 5.49. We get:

  $P^*[\tilde{M}_t\le -k, Z_T=-k+l] = \left(\frac{p^*}{1-p^*}\right)^l P^*[Z_T=-k-l] = \left(\frac{1-p^*}{p^*}\right)^k P^*[Z_T=k+l]$.

- $(\mathrm{I}) = \sum_{l\ge1}(K-S_0\hat{b}^{-k+l})\left(\frac{1-p^*}{p^*}\right)^k P^*[Z_T=k+l] = \left(\frac{1-p^*}{p^*}\right)^k \hat{b}^{-2k}\underbrace{\sum_{l\ge1}(\hat{b}^{2k}K - S_0\hat{b}^{k+l})_+ P^*[Z_T=k+l]}_{=:(\mathrm{II})}$

- Define $\tilde{K} := \hat{b}^{2k}K = \left(\frac{S_0}{B}\right)^2 K$, $\tilde{B} := \frac{S_0^2}{B} = S_0\hat{b}^k$.

- $(\mathrm{II}) = \sum_{l\ge1} E^*[(\tilde{K}-S_0\hat{b}^{k+l})_+\mathbb{1}_{Z_T=k+l}] = \sum_{l\ge1}E^*[(\tilde{K}-S_T)_+\mathbb{1}_{Z_T=k+l}] = E^*[(\tilde{K}-S_T)_+\sum_{l\ge1}\mathbb{1}_{Z_T=k+l}]$

  $= E^*[(\tilde{K}-S_T)_+\mathbb{1}_{Z_T>k}]$. ✓

- $S_T > \tilde{B} = S_0 \hat{b}^{k} \iff S_0 \hat{b}^{Z_T} > S_0 \hat{b}^{k} \iff Z_T > k$.
- $(II) = E^*\left[(\tilde{K} - S_T)_+ \mathbb{1}_{S_T > \tilde{B}} \right]$
- $(I) = \left(\frac{1-p^*}{p^*} \right)^k \left(\frac{B}{S_0} \right)^2 E^*\left[(\tilde{K} - S_T)_+ \mathbb{1}_{S_T > \tilde{B}} \right]$

$$\Rightarrow \pi(C^{put}_{d\&i}) = \frac{1}{(1+r)^T} \left\{ E^*\left[(K - S_T)_+ \mathbb{1}_{S_T \le B} \right] + \left(\frac{1-p^*}{p^*} \right)^k \left(\frac{B}{S_0} \right)^2 E^*\left[(\tilde{K} - S_T)_+ \mathbb{1}_{S_T > \tilde{B}} \right] \right.$$

(c) Lookback put option: $C^{put}_{max} = \max_{0 \le t \le T} S_t - S_T$.

- $\pi(C^{put}_{max}) = E^*\left[\frac{C^{put}_{max}}{(1+r)^T} \right] = \frac{1}{(1+r)^T} E^*\left[\max_{0 \le t \le T} S_t \right] - E^*\left[\frac{S_T}{(1+r)^T} \,\middle|\, F_0 \right]$

$$= \frac{1}{(1+r)^T} E^*\left[\max_{0 \le t \le T} S_t \right] - S_0.$$

- We have $S_t = S_0 \hat{b}^{Z_t}$. Define $M_t := \max_{0 \le s \le t} Z_s$, $0 \le M_t \le t$ (bc. $Z_0 = 0$).
- $\max_{0 \le t \le T} S_t = \max_{0 \le t \le T} S_0 \hat{b}^{Z_t} = S_0 \hat{b}^{\max_{0 \le t \le T} Z_t} = S_0 \hat{b}^{M_T}$.
- $E^*\left[\max_{0 \le t \le T} S_t \right] = E^*\left[S_0 \hat{b}^{M_T} \right] = E^*\left[S_0 \sum_{k=0}^{T} \hat{b}^k \mathbb{1}_{M_T = k} \right] = S_0 \sum_{k=0}^{T} \hat{b}^k P^*[M_T = k]$.
- By the reflection principle for P^* (FS, Lem. 5.48):

$$P^*[M_T = k] = E^*[\mathbb{1}_{M_T = k}] = E^*\left[\sum_{l \ge 0} \mathbb{1}_{M_T = k} \mathbb{1}_{Z_T = k-l} \right] = \sum_{l \ge 0} P^*[M_T = k, Z_T = k-l]$$

$$= \sum_{l \ge 0} \frac{1}{1-p^*} \left(\frac{p^*}{1-p^*} \right)^k \frac{k+l+1}{T+1} P^*[Z_{T+1} = -1-k-l]$$

$$= \frac{1}{1-p^*} \left(\frac{p^*}{1-p^*} \right)^k \frac{1}{T+1} E^*\left[\sum_{l \ge 0} (k+1+l) \mathbb{1}_{Z_{T+1} = -k-1-l} \right]$$

$$= \text{———————} \;\text{"}\; \text{————} \; E^*\left[-Z_{T+1} \mathbb{1}_{Z_{T+1} \le -k-1} \right]$$

$$\Rightarrow \pi(C^{put}_{max}) = \frac{S_0}{(1+r)^T (1-p^*)(T+1)} \sum_{k=0}^{T} \hat{b}^k \left(\frac{p^*}{1-p^*} \right)^k E^*\left[-Z_{T+1} \mathbb{1}_{Z_{T+1} \le k+1} \right] - S_0.$$

(d) Lookback call option: $C^{call}_{min} = S_T - \min_{0 \le t \le T} S_t$.

- $\pi(C^{call}_{min}) = E^*\left[\frac{C^{call}_{min}}{(1+r)^T} \right] = E^*\left[\frac{S_T}{(1+r)^T} \,\middle|\, F_0 \right] - \frac{1}{(1+r)^T} E^*\left[\min_{0 \le t \le T} S_t \right]$

$$= S_0 - \frac{1}{(1+r)^T} E^*\left[\min_{0 \le t \le T} S_t \right].$$

- We have $S_t = S_0 \hat{b}^{Z_t}$. Define $\tilde{M}_t := \min_{0 \le s \le t} Z_s$, $-t \le \tilde{M}_t \le 0$ (bc. $Z_0 = 0$).
- $\min_{0 \le t \le T} S_t = \min_{0 \le t \le T} S_0 \hat{b}^{Z_t} = S_0 \hat{b}^{\min_{0 \le t \le T} Z_t} = S_0 \hat{b}^{\tilde{M}_t}$.
- $E^*\left[\min_{0 \le t \le T} S_t \right] = E^*\left[S_0 \hat{b}^{\tilde{M}_T} \right] = E^*\left[S_0 \sum_{k=0}^{-T} \hat{b}^k \mathbb{1}_{\tilde{M}_T = k} \right] = S_0 \sum_{k=0}^{T} \hat{b}^{-k} P^*[\tilde{M}_T = -k]$.
- By the analogous reflection principle for $P^*[\tilde{M}_T = -k]$:

$$P^*[\tilde{M}_T = -k] = E^*[\mathbb{1}_{\tilde{M}_T = -k}] = E^*\left[\sum_{l \ge 0} \mathbb{1}_{\tilde{M}_T = -k} \mathbb{1}_{Z_T = -k+l} \right] = \sum_{l \ge 0} P^*[\tilde{M}_T = -k, Z_T = -k+l]$$

$$= \frac{1}{1-p^*} \left(\frac{1-p^*}{p^*} \right)^k \frac{-k-l-1}{T+1} P^*[Z_{T+1} = k+l+1]$$

$$= \frac{1}{1-p^*} \left(\frac{1-p^*}{p^*} \right)^k \frac{1}{T+1} E^*\left[\sum_{l \ge 0} (-k-1-l) \mathbb{1}_{Z_{T+1} = k+1+l} \right]$$

$$= \text{—————} \;\text{"}\; \text{——————} \; E^*\left[-Z_{T+1} \mathbb{1}_{Z_{T+1} \ge k+1} \right]$$

$$\Rightarrow \pi(C^{put}_{min}) = \frac{S_0}{(1+r)^T (1-p^*)(T+1)} \sum_{k=0}^{T} \hat{b}^{-k} \left(\frac{1-p^*}{p^*} \right)^k E^*\left[-Z_{T+1} \mathbb{1}_{Z_{T+1} \ge k+1} \right] - S_0.$$