IMF Serie 6 — Florian Krach

Exercise 6.1

Notation: $P^{T+\delta} = Q^{T+\delta}$, $P^A = Q^A$

(a) Recall and prove the *numeraire change theorem*.

(b) Show that the forward Libor rate $(L(t,T,T+\delta))_t$[1] is a martingale under the *forward probability* $P^{T+\delta}$. Derive the arbitrage free price of a caplet with payoff $C^{caplet} = \delta(L(t,T,T+\delta)-K)_+$.

(c) Using the formula

$$S_{T_0,T_N}(t) = \frac{P(t,T_0)-P(t,T_N)}{\sum_{k=0}^N \delta P(t,T_k)}$$

show that the swap rate $(S_{T_0,T_N}(t))_t$ is a martingale under the *annuity probability* P^A where the *annuity* is $A(t) = \sum_{k=0}^N \delta P(t,T_k)$. Derive the arbitrage free price of a swaption with payoff $C^{swaption} = A(T_f)(S_{T_0,T_N}(t)-K)_+$.

[1] the forward zero-coupon $P(t,T,T+\delta) = \frac{1}{1+\delta L(t,T,T+\delta)}$ and $P(t,T,T+\delta) = \frac{P(t,T+\delta)}{P(t,T)}$.

a) Change of numeraire Thm: $\mathcal{M}_e(S^1) = \{\tilde{Q} \mid \frac{d\tilde{Q}}{dQ} = \frac{X_T^1}{X_0^1},\ \text{for } Q \in \mathcal{M}_e(S^0)\}$ where $X_t^d = \frac{S_t^d}{S_t^0}$

Proof: "$\supset$": $\rightsquigarrow$ take $\tilde{Q}$ defined by $\frac{d\tilde{Q}}{dQ} = \frac{X_T^1}{X_0^1}$, for $Q \in \mathcal{M}_e(S^0)$

$\rightsquigarrow$ $\tilde{Q} \ll Q$ holds by def. of $\tilde{Q}$

$\rightsquigarrow$ since $S_t^0 > 0$, $S_t^1 > 0$ $\forall t$ $\Rightarrow$ $\frac{X_T^1}{X_0^1} = \left(\frac{S_0^1}{S_0^0}\right)^{-1}\left(\frac{S_T^1}{S_T^0}\right) > 0$

Claim 1: $Q \ll \tilde{Q}$ — Proof 1: let $A \in \mathcal{F}$ with $\tilde{Q}(A) = 0$

$\Leftrightarrow$ $0 = \tilde{Q}(A) = \int_A \underbrace{\frac{X_T^1}{X_0^1}}_{>0} dQ$ $\rightsquigarrow$ if $Q(A) > 0$ $\Rightarrow$ also $\int_A \frac{X_T^1}{X_0^1} dQ > 0$ has to hold (see Measure Theory) ↯ □ (Claim 1)

$\Rightarrow$ $\tilde{Q} \sim Q \sim P$ $\Rightarrow$ $\tilde{Q} \sim P$, so $\tilde{Q}$ is an equivalent measure

$\rightsquigarrow$ $\tilde{Q}(\Omega) = \mathbb{E}_Q\left(\frac{X_T^1}{X_0^1}\right) = \mathbb{E}_Q\left(\mathbb{E}_Q\left(\frac{X_T^1}{X_0^1} \mid \mathcal{F}_0\right)\right) \overset{X_T^1 \text{ is } Q\text{-mart.}}{=} \mathbb{E}_Q\left(\frac{X_0^1}{X_0^1}\right) = 1$ $\Rightarrow$ $\tilde{Q}$ is equiv. prob.-meas.

Claim 2: $\left(\frac{S_t^i}{S_t^1}\right)_{t\geq 0}$ is a $\tilde{Q}$-martingale $\forall\, 0 \leq i \leq d$ $\rightsquigarrow$ with Claim 2 we have that $\tilde{Q} \in \mathcal{M}_e(S^1)$

Proof 2: $\rightsquigarrow$ $0 \leq r \leq t \leq T$ $\rightsquigarrow$ $\mathbb{E}_{\tilde{Q}}\left(\frac{S_t^i}{S_t^1} \mid \mathcal{F}_r\right) \overset{\text{Bayes formula}}{=} \left(\mathbb{E}_Q\left(\frac{d\tilde{Q}}{dQ} \mid \mathcal{F}_r\right)\right)^{-1} \cdot \mathbb{E}_Q\left(\frac{S_t^i}{S_t^1} \cdot \mathbb{E}_Q\left(\frac{d\tilde{Q}}{dQ} \mid \mathcal{F}_t\right) \mid \mathcal{F}_r\right) = \left(\mathbb{E}_Q\left(\frac{X_T^1}{X_0^1} \mid \mathcal{F}_r\right)\right)^{-1} \cdot$

$\cdot\, \mathbb{E}_Q\left(\frac{S_t^i}{S_t^1} \cdot \mathbb{E}_Q\left(\frac{X_T^1}{X_0^1} \mid \mathcal{F}_t\right) \mid \mathcal{F}_r\right) \overset{X_T\ Q\text{-mart.}}{=} \left(\frac{X_r^1}{X_0^1}\right)^{-1} \cdot \mathbb{E}_Q\left(\frac{S_t^i}{S_t^1} \cdot \frac{X_t^1}{X_0^1} \mid \mathcal{F}_r\right) = \left(\frac{X_0^1}{X_r^1}\right) \cdot \mathbb{E}_Q\left(\frac{S_t^i}{S_t^1}\left(\frac{S_0^1}{S_0^0}\right)^{-1}\left(\frac{S_t^1}{S_t^0}\right) \mid \mathcal{F}_r\right) =$

$= \frac{S_0^1}{S_0^0} \cdot \frac{S_r^0}{S_r^1} \cdot \underbrace{\mathbb{E}_Q\left(\frac{X_t^i}{X_0^1} \mid \mathcal{F}_r\right)}_{= \frac{X_r^i}{X_0^1}} = \frac{S_0^0}{S_0^1} \cdot \frac{S_r^0}{S_r^1} \cdot \frac{S_r^i}{S_r^0} \cdot \frac{S_0^0}{S_0^1} = \frac{S_r^i}{S_r^1}$ □ (Claim 2) □ ("$\supset$")

"$\subset$": $\rightsquigarrow$ using "$\supset$" and exchanging S^1 & S^0 we have: $\{Q \mid \frac{dQ}{d\tilde{Q}} = \frac{Y_T^0}{Y_0^0},\ \tilde{Q} \in \mathcal{M}_e(S^1)\} \subset \mathcal{M}_e(S^0)$, for $Y_t^i = \frac{S_t^i}{S_t^1}$

$\Rightarrow$ $\forall \tilde{Q} \in \mathcal{M}_e(S^1)$: $\exists Q \in \mathcal{M}_e(S^0)$: $\frac{dQ}{d\tilde{Q}} = \frac{S_T^0}{S_T^1} \cdot \frac{S_0^1}{S_0^0} \overset{\substack{\text{use that } Q\sim\tilde{Q} \\ (\text{see "}\supset\text{"})}}{\underset{(*)}{\Rightarrow}} \frac{d\tilde{Q}}{dQ} = \frac{S_T^1}{S_T^0} \cdot \frac{S_0^0}{S_0^1} = \frac{X_T^1}{X_0^1}$ $\Rightarrow$ $\tilde{Q} \in$ RHS □ ("$\subset$") ■

Proof (*): we have $\frac{dQ}{dQ} = 1$, by Radon-Nikodym Thm, we know that for $\nu \ll \tau \ll \lambda$ meas. it holds that $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\tau} \cdot \frac{d\tau}{d\lambda}$

(Proof: $\nu(A) = \int_A \left(\frac{d\nu}{d\lambda}\right) d\lambda$, by R-N, $\int_A \frac{d\nu}{d\tau} \cdot \underbrace{\frac{d\tau}{d\lambda} d\lambda}_{= d\tau,\ \text{by R-N}} = \int_A \underbrace{\frac{d\nu}{d\tau} \cdot d\tau}_{= d\nu,\ \text{by R-N}} = \nu(A)$ $\rightsquigarrow$ since it holds $\forall A$ mbl $\Rightarrow$ □)

$\rightsquigarrow$ using this we have, since $Q \sim \tilde{Q}$: $1 = \frac{dQ}{dQ} = \frac{dQ}{d\tilde{Q}} \cdot \frac{d\tilde{Q}}{dQ}$ $\Leftrightarrow$ $\frac{dQ}{d\tilde{Q}} = \left(\frac{d\tilde{Q}}{dQ}\right)^{-1}$ (and also the other way)

$\rightsquigarrow$ this is well defined since $Q \sim \tilde{Q}$ implies that $\frac{d\tilde{Q}}{dQ} > 0$, $\frac{dQ}{d\tilde{Q}} > 0$ P-a.s. □ (*)

b) Proof: $\rightsquigarrow$ the forward Libor rate is $(L(t,T,T+\delta))_t = \left[\frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)\right]_t$

$\rightsquigarrow$ the forward probability $Q^{T+\delta}$ is a measure $\in \mathcal{M}_e(P(\cdot,T+\delta))$ $\rightsquigarrow$ by Change of numeraire Thm such a meas. exists, since $P(\cdot,T+\delta) > 0$ and can therefore be used as numeraire

$\rightsquigarrow$ our market is given by $(B_t, P(t,T_1), \dots, P(t,T_N))_{t\in[T_0,\dots,T_N]}$, where $T_0 < T_1 < \dots < T_N$, $\rightsquigarrow$ $T, T+\delta \in \{T_0,\dots,T_N\}$

$\rightsquigarrow$ if we want we can assume (by change of numeraire Thm) wlog. that $B_0 = 1$ (by replacing $(B_t)_t$ by $\left(\frac{B_t}{B_0}\right)_t$ as new numeraire)

$\rightsquigarrow$ since $Q^{T+\delta} \in \mathcal{M}_e(P(\cdot,T+\delta))$ we have that $P(\cdot,T+\delta)$ is our numeraire and $\forall \tilde{T} \in [T_1,\dots,T_N]$: $\left(\frac{P(t,\tilde{T})}{P(t,T+\delta)}\right)_t$ is a $Q^{T+\delta}$-martingale

$\Rightarrow$ i.e.: $\left(\frac{P(t,T)}{P(t,T+\delta)}\right)_t$ is a $Q^{T+\delta}$-martingale $\Rightarrow$ also $(L(t,T,T+\delta))_t$ is a $Q^{T+\delta}$-martingale ✓

$\rightsquigarrow$ the arbitrage free price of C_t^{caplet}, (a payoff at time $T+\delta$) is given by (using that $Q \in \mathcal{M}_e((B_t)_t)$)

$$\mathbb{E}_Q\left(\frac{C_t^{caplet}}{B_{T+\delta}}\right) = \mathbb{E}_{Q^{T+\delta}}\left(\frac{C_t^{caplet}}{B_{T+\delta}} \cdot \frac{dQ}{dQ^{T+\delta}}\right) = \mathbb{E}_{Q^{T+\delta}}\left(\frac{C_t^{caplet}}{B_{T+\delta}} \cdot \frac{B_{T+\delta}}{P(T+\delta,T+\delta)} \cdot \frac{P(0,T+\delta)}{B_0}\right) = \frac{P(0,T+\delta)}{B_0} \mathbb{E}_{Q^{T+\delta}}\left(C_t^{caplet}\right) =$$

($\frac{dQ^{T+\delta}}{dQ} \overset{a)}{=} \frac{P(T+\delta,T+\delta)}{B_{T+\delta}} \cdot \left(\frac{P(0,T+\delta)}{B_0}\right)^{-1}$ & (*); $P(T+\delta,T+\delta) = 1$)

$$= \frac{P(0,T+\delta)}{B_0} \mathbb{E}_{Q^{T+\delta}}\left[\delta\left(L(t,T,T+\delta)-K\right)^+\right]$$

c.) **Proof**: $\rightsquigarrow$ since $\forall 0 \le k \le N$: $P(t, T_k)$ is an asset in the market, we can extend our market s.t. also $A(t) = \sum_{k=0}^{N} \delta\, P(t, T_k)$ is an asset, without loosing (NA), (since we can't generate new value processes)

$\rightsquigarrow$ since $P(\cdot, T_k) > 0\ \forall k \Rightarrow A(\cdot) > 0 \Rightarrow$ by Change of numeraire Thm, we can chose $A(t)$ as numeraire and we find a meas. $Q^A \in \mathcal{M}_e(A)$

$\rightsquigarrow$ under Q^A we have that: $\forall k$: $\left(\frac{P(t,T_k)}{A(t)}\right)_t$ is a Q^A-martingale

$\Rightarrow$ also $\left(S_{T_0,T_N}(t)\right)_t = \left(\frac{P(t,T_0)}{A(t)} - \frac{P(t,T_N)}{A(t)}\right)_t$ is a Q^A-martingale, as sum of 2 Q^A-mart. ✓

$\rightsquigarrow$ the arbitrage-free price of $C_t^{sw.}$ (a payoff at time T_f) is given by (using again the (EMM) $Q \in \mathcal{M}_e((B_t)_t)$:

$$\mathbb{E}_Q\left(\frac{C_t^{sw.}}{B_{T_f}}\right) \overset{\text{as in b)}}{=} \mathbb{E}_{Q^A}\left(\frac{C_t^{sw}}{B_{T_f}} \cdot \frac{dQ}{dQ^A}\right) \overset{\text{see b)}}{=} \mathbb{E}_{Q^A}\left(\frac{C_t^{sw}}{B_{T_f}} \cdot \frac{B_{T_f}}{A(T_f)} \cdot \frac{A(0)}{B_0}\right) = \frac{A(0)}{B_0} \cdot \mathbb{E}_{Q^A}\left[\frac{A(T_f)}{A(T_f)} \cdot (S_{T_0,T_N}(t) - K)^+\right] =$$

$$= \frac{A(0)}{B_0} \mathbb{E}_{Q^A}\left[(S_{T_0,T_N}(t) - K)^+\right]$$

Exercise 6.2
Derive a formula for the arbitrage free price of following contingent claims.

(a) *up-and-out call option*, $C_{u\&o}^{call} = (S_T - K)1_{\max_{0\le t\le T} S_t < B}$

(b) *down-and-in put option*, $C_{d\&i}^{put} = (K - S_T)1_{\min_{0\le t\le T} S_t \le B}$

(c) *lookback put option*, $C_{max}^{put} = \max_{0\le t\le T} S_t - S_T$

(d) *lookback call option*, $C_{min}^{call} = S_T - \min_{0\le t\le T} S_t$

$\rightsquigarrow$ we assume to have an Binomial model $((1+r)^t, S_t)_{t=0,..,T}$, with $S_t = S_0 \hat{b}^{Z_t}$ with $Z_t = \sum_{i=1}^{t} y_i$, where $\omega \in \Omega = \{-1,+1\}^T$, $\omega = (y_1, .., y_T)$

$\rightsquigarrow$ let Q be the EMM with $q = Q(y_i = 1)$, $(1-q) = Q(y_i = -1)$

$\rightsquigarrow Q(Z_t = k) = \begin{cases} q^{\frac{t+k}{2}} (1-q)^{\frac{t-k}{2}} \cdot \binom{t}{\frac{t+k}{2}} & \text{, if } t+k \text{ even} \\ 0 & \text{, else} \end{cases}$

$\rightsquigarrow$ let $\tilde{M}_t := \max_{0\le s\le t} S_s$, $M_t := \max_{0\le s\le t} Z_s$, $\tilde{N}_t := \min_{0\le s\le t} S_s$, $N_t := \min_{0\le s\le t} Z_s$

a) **sol**: the arbitrage free price of $C_{u\&o}^{call} =: C$ is given by $\mathbb{E}_Q\left(\frac{C}{(1+r)^T}\right) = \frac{1}{(1+r)^T} \cdot \mathbb{E}_Q(C)$

$\rightsquigarrow$ since we have that $\mathbb{E}_Q(C_{u\&o}^{call}) + \mathbb{E}_Q(C_{u\&i}^{call}) = \frac{1}{(1+r)^T} \mathbb{E}_Q\left((S_T - K)^+ \left(1\!\!1_{\max_{0\le t\le T} S_t < B} + 1\!\!1_{\max_t S_t \ge B}\right)\right) = \frac{1}{(1+r)^T} \mathbb{E}_Q\left((S_T - K)^+\right)$

$\Rightarrow \mathbb{E}_Q(C_{u\&o}^{call}) = \mathbb{E}_Q\left(\frac{(S_T - K)^+}{(1+r)^T}\right) - \mathbb{E}_Q(C_{u\&i}^{call})$

$\rightsquigarrow \mathbb{E}_Q(C_{u\&i}^{call})$ can be computed as done in lecture $\Rightarrow \mathbb{E}_Q\left(\frac{C_{u\&i}^{call}}{(1+r)^T}\right) = \frac{1}{(1+r)^T} \cdot \left(\mathbb{E}_Q\left[(S_T - K)^+ \cdot 1\!\!1_{S_T \ge B}\right] + \left(\frac{q}{1-q}\right)^k \left(\frac{B}{S_0}\right)^2 \cdot \right.$ $\left. \cdot \mathbb{E}_Q\left[(S_T - \tilde{K})^+ 1\!\!1_{S_T < \tilde{B}}\right]\right)$, where $B \overset{\text{wlog}}{\in} \{S_0 \cdot \hat{b}^v \mid v \in \{T, T-2, .., -T\}\}$ and k s.t. $B = S_0 \hat{b}^k$, $\tilde{K} = K \cdot \hat{b}^{-2k} = K\left(\frac{S_0}{B}\right)^2$, $\tilde{B} := \frac{S_0^2}{B}$

$\rightsquigarrow \mathbb{E}_Q\left(\frac{(S_T - K)^+}{(1+r)^T}\right) = \frac{1}{(1+r)^T} \cdot \sum_{n=0}^{T} (S_0 \cdot \hat{b}^{T-2n} - K)^+ \cdot q^{T-n} (1-q)^n \binom{T}{T-n}$, same can be done for $\mathbb{E}_Q(C_{u\&i}^{call})$

b.) **sol**: $\rightsquigarrow$ the arbitrage free price for $C_{d\&i}^{put}$ is given by: $\mathbb{E}_Q\left(\frac{C_{d\&i}^{put}}{(1+r)^T}\right) = \frac{1}{(1+r)^T} \mathbb{E}_Q\left((K - S_T)^+ \cdot 1\!\!1_{\tilde{N}_T \le B}\right)$

$\rightsquigarrow$ wlog: $B \in \{S_0 \hat{b}^v \mid v \in \{-T, -T+2, .., T\}\}$ $\rightsquigarrow$ let $k \in \mathbb{N}$ s.t. $B = S_0 \hat{b}^k$

$\rightsquigarrow \mathbb{E}_Q\left((K - S_T)^+ 1\!\!1_{\tilde{N}_T \le B}\right) = \mathbb{E}_Q\left((K - S_T)^+ 1\!\!1_{S_T \le B}\right) + \mathbb{E}_Q\left((K - S_T)^+ 1\!\!1_{S_T > B} 1\!\!1_{\tilde{N}_T \le B}\right)$

$\rightsquigarrow \mathbb{E}_Q\left((K - S_T)^+ 1\!\!1_{S_T \le B}\right) = \sum_{j \in \{-T, -T+2, .., k\}} (K - S_0 \hat{b}^j)^+ \cdot Q(Z_T = j) = \sum_{j \in \{-T, -T+2, .., k\}} (K - S_0 \hat{b}^j) \cdot q^{\frac{T+j}{2}} (1-q)^{\frac{T-j}{2}} \binom{T}{\frac{T+j}{2}}$

$\rightsquigarrow \mathbb{E}_Q\left((K - S_T)^+ 1\!\!1_{S_T > B} 1\!\!1_{\tilde{N}_T \le B}\right) = \sum_{l \ge 1} \mathbb{E}_Q\left((K - S_0 \hat{b}^{Z_T})^+ 1\!\!1_{Z_T = k+l} 1\!\!1_{N_T \le k}\right) = \sum_{l \ge 1} (K - S_0 \hat{b}^{k+l})^+ \cdot Q(Z_T = k+l, N_T \le k) =: (I)$

$\rightsquigarrow Q(Z_T = k+l, \min_t Z_T \le k) =$ ↙ reflection principle holds equivalently as in Lem 5.49 $= \left(\frac{1-q}{q}\right)^{-l} Q(Z_T = k-l) = \left(\frac{1-q}{q}\right)^{-l} \cdot$

$\cdot q^{\frac{T+k-l}{2}} (1-q)^{\frac{T-k+l}{2}} \binom{T}{\frac{T+k-l}{2}} = \left(\binom{T}{\frac{T-k+l}{2}} q^{\frac{T-k+l}{2}} (1-q)^{\frac{T+k-l}{2}}\right) \cdot q^{k-l} (1-q)^{l-k} (1-q)^{-l} q^{l} = Q(Z_t = -k+l) \cdot \left(\frac{q}{1-q}\right)^k$

$\Rightarrow (I) = \sum_{l \ge 1} (K - S_0 \hat{b}^{k+l})^+ \cdot Q(Z_T = -k+l) \left(\frac{q}{1-q}\right)^k = \left(\frac{q}{1-q}\right)^k \sum_{l \ge 1} \hat{b}^{2k} (\overbrace{K \hat{b}^{-2k}}^{=: \hat{K}} - S_0 \hat{b}^{-k+l})^+ \cdot$

$\cdot Q(Z_t = -k+l) = \left(\frac{q}{1-q}\right)^k \hat{b}^{2k} \cdot \mathbb{E}_Q\left((\hat{K} - S_0 \hat{b}^{Z_T})^+ 1\!\!1_{Z_T > -k}\right) = \left(\frac{q}{1-q}\right)^k \underbrace{\hat{b}^{2k}}_{= (B/S_0)^2} \cdot \mathbb{E}_Q\left((\hat{K} - S_T)^+ 1\!\!1_{S_T > \underbrace{S_0 \cdot \hat{b}^{-k}}_{= B \cdot \hat{b}^{-2k} = \frac{S_0^2}{B} =: \hat{B}}}\right) =$

$= \left(\frac{q}{1-q}\right)^k \hat{b}^{2k} \sum_{j \in \{-k+2, -k+4, .., T\}} (\hat{K} - S_0 \hat{b}^j)^+ \cdot q^{\frac{T+j}{2}} (1-q)^{\frac{T-j}{2}} \cdot \binom{T}{\frac{T+j}{2}}$

$\rightsquigarrow$ combining those calculations yields the formula

c.), d.) $\rightsquigarrow$ see book Exa 5.53

Exercise 6.3 We consider a binomial market model with N periods on a period of time of length T. The riskless asset grows at a rate $r = \frac{RT}{N}$, where R is the (constant) instantaneous interest rate, and the risky asset's price goes up by a factor $1+u$ and down by a factor $1+d$ such that

$$\log\left(\frac{1+u}{1+r}\right) = -\log\left(\frac{1+d}{1+r}\right) = \sigma\sqrt{\frac{T}{N}},$$

for some constant σ. The starting values (at time t=0) of both the assets is 1 $\mathbb{P}$-a.s. The unique equivalent martingale measure $\mathbb{P}^*$ for S^1 is such that the $(Y_i)_{i\in\{1,2,\ldots,N\}}$ are i.i.d. and given by

$$\mathbb{P}^*\left[Y_i = 1+d\right] = 1 - \mathbb{P}^*\left[Y_i = 1+u\right] = \frac{u-r}{u-d} = p^*$$

We study the limiting case for $N\to\infty$.

(a) Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables of the form :

$$Z_n = \sum_{i=1}^{n} X_i^n$$

for $n\in\mathbb{N}$, $X_i^n \in \{-\sigma\sqrt{\frac{T}{n}}, \sigma\sqrt{\frac{T}{n}}\}$ and the variables $(X_i^n)_{i\in\{1,2,\ldots,n\}}$ are independent identically distributed with mean μ_n. The constants μ_n are such that $\lim_{n\to\infty} n\mu_n = \mu$.

Prove that the sequence $(Z_n)_{n\in\mathbb{N}}$ converges in law to a gaussian random variable with mean μ and variance $\sigma^2 T$.

Hint: Use the fact that point-wise convergence of the characteristic functions of a sequence of random variables (if the limiting function ϕ is continuous at 0) implies the convergence in law of this sequence of random variables to a random variable whose characteristic function is ϕ.

(b) We consider a European put option, with strike K and maturity T. Show that its value at time 0 is given by

$$V_0^{P,N} = \mathbb{E}^*\left[\left(\frac{K}{(1+r)^N} - S_0^1\exp\left(Z_N\right)\right)^+\right],$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$, and Z_N is a random variable that you will define.

(c) Use part a) to prove the following asymptotic price :

$$\lim_{N\to\infty} V_0^{P,N} = Ke^{-RT}\Phi(-d_2) - S_0^1\Phi(-d_1),$$

where $d_1 = \dfrac{\log\left(\frac{S_0}{K}\right) + RT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$ and Φ is the cumulative distribution function of a standard normal random variable.

Hint: Use the value of p^* and of u to prove that $\lim_{N\to\infty} N\mathbb{E}^*\left[\log\left(\frac{Y_i}{1+r}\right)\right] = -\frac{\sigma^2 T}{2}$

Notation: $S_t := S_t^1$, $B_t := S_t^0$

a) Proof: ⇝ use Lindeberg-Feller Thm (Thm 2.24 Prob. Th.)

⇝ define $\tilde{X}_i^n := X_i^n - \mu_n \Rightarrow E(\tilde{X}_i^n) = 0$

⇝ $Var(\tilde{X}_i^n) = Var(X_i^n) = E((X_i^n)^2) - E(X_i^n)^2 = \left(\sigma^2\frac{T}{n}p + \sigma^2\frac{T}{n}(1-p)\right) - \mu_n^2 = \sigma^2\frac{T}{n} - \mu_n^2$

$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^n E(\tilde{X}_i^{n\,2}) = \lim_{n\to\infty} n\cdot\left(\sigma^2\frac{T}{n} - \mu_n^2\right) = \sigma^2 T - \lim_{n\to\infty} n\mu_n^2 = \sigma^2 T$, (since $n\mu_n \xrightarrow{n\to\infty} \mu \Rightarrow \mu_n \xrightarrow{n\to\infty} 0 \Rightarrow n\mu_n^2 = n\mu_n\mu_n \xrightarrow{n\to\infty} \mu\cdot 0 = 0$)

⇝ $\forall \varepsilon>0$ choose n_0 so big such that: $\sigma\sqrt{\frac{T}{n_0}} < \frac{\varepsilon}{2}$, $\mu_n < \frac{\varepsilon}{2}$ $\forall n\geq n_0$

$\Rightarrow \forall n\geq n_0$: $|\tilde{X}_i^n| = \sigma\sqrt{\frac{T}{n}} + |\mu_n| < \varepsilon$

$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^n E\left(\tilde{X}_i^{n\,2}\,\mathbb{1}_{\{|\tilde{X}_i^n|>\varepsilon\}}\right) = 0$

⇝ conditions for Lindeberg-Feller fulfilled ⇒

$\sum_{i=1}^n \tilde{X}_i^n \xrightarrow[n\to\infty]{d} N(0,\sigma^2 T)$

$\Leftrightarrow \lim_{n\to\infty} Z_n - \mu = \lim_{n\to\infty}\left(\sum_{i=1}^n X_i^n - n\mu_n\right) \overset{d}{=} N(0,\sigma^2 T)$

$\Leftrightarrow \lim_{n\to\infty} Z_n \overset{d}{=} N(\mu,\sigma^2 T)$ ■

b) Proof: ⇝ $C^{put} = (K - S_N)^+$

⇝ its price is given (using $Q = P^* \in \mathcal{M}el((B_t)_t)$) by

$E^*\left(\frac{C^{put}}{B_N}\right) = E^*\left(\frac{1}{(1+r)^N}\cdot(K-S_N)^+\right) = E^*\left(\left(\frac{K}{(1+r)^N} - \frac{S_N}{(1+r)^N}\right)^+\right)$ (1)

⇝ let $R_t = \begin{cases} u, & \text{if price goese up from time t-1 to t} \\ d, & \text{else}\end{cases}$ as defined in lecture ⇒ $S_N = S_0\cdot\prod_{t=1}^N(1+R_t) = (1+r)^N\cdot S_0\cdot\prod_{t=1}^N\left(\frac{1+R_t}{1+r}\right) =$
$= (1+r)^N\cdot S_0\cdot\prod_{t=1}^N\exp\left(\log\left(\frac{1+R_t}{1+r}\right)\right)$ ⇝ define $X_t := \log\left(\frac{1+R_t}{1+r}\right)$ ⇝ let $Z_t := \sum_{i=1}^t X_i$ ⇝ we have $1+R_t = Y_t$

$\Rightarrow S_N = (1+r)^N\cdot S_0\cdot\prod_{t=1}^N\exp(X_t) = (1+r)^N\cdot S_0\cdot\exp\left(\sum_{t=1}^N X_t\right) = (1+r)^N\cdot S_0\cdot\exp(Z_N)$ (2)

$\Rightarrow V_0^{P,N} = E^*\left(\frac{C^{put}}{B_N}\right) \overset{(1),(2)}{=} E^*\left[\left(\frac{K}{(1+r)^N} - S_0\cdot\exp(Z_N)\right)^+\right]$ ■

c) Proof: $\lim_{N\to\infty} V_0^{P,N} \overset{b)}{=} \lim_{N\to\infty} E^*\left(\left(\frac{K}{(1+r_N)^N} - S_0\exp(Z_N)\right)^+\right)$ (6)

⇝ since $S_N = S_0\exp(Z_N)\geq 0$ (for $S_0>0$) and $\lim_{N\to\infty}(1+r_N)^N = \lim_{N\to\infty}\left(1+\frac{RT}{N}\right)^N = e^{RT}\in(0,\infty)$ ⇒ $\left(\frac{1}{(1+r_N)^N}\right)_{N\in\mathbb{N}}$ is bounded (at least when starting from some $N_0\in\mathbb{N}$ big enough)

$\Rightarrow 0\leq\left(\frac{K}{(1+r_N)^N} - S_0\exp(Z_N)\right)^+ \leq \frac{K}{(1+r_N)^N} \leq M$ $\forall N\geq N_0$, where $M<\infty$ is some const. ⇒ the integrand can be dominated (5)

⇝ by assumption on our market we have that $Z_N = \sum_{i=1}^N X_i^N$ with $X_i^N = \log\left(\frac{1+R_i}{1+r}\right)\in\left\{\pm\sigma\sqrt{\frac{T}{N}}\right\}$, with $(X_i^N)_{i=1,\ldots,N}$ iid (since R_i are iid) ⇝ $E(X_1^N) = -\sigma\sqrt{\frac{T}{N}}\cdot p^* + \sigma\sqrt{\frac{T}{N}}(1-p^*) = \sigma\sqrt{\frac{T}{N}} - 2p^*\sqrt{\frac{T}{N}}\sigma =: \mu_N$ ($p^*\in[0,1]$) ⇝ obviously also u, d, p^* depend on N

⇝ $\lim_{N\to\infty} N\cdot\mu_N = \lim_{N\to\infty}\sigma\sqrt{TN} - 2p^*\cdot\sigma\sqrt{TN} = (I)$ ⇝ $p_N^* = \frac{u_N - r_N}{u_N - d_N}$

⇝ $\log\left(\frac{1+u_N}{1+r_N}\right) = \sigma\sqrt{\frac{T}{N}} \Leftrightarrow \frac{1+u_N}{1+r_N} = \exp\left(\sigma\sqrt{\frac{T}{N}}\right) \Leftrightarrow u_N = (1+r_N)\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - 1$ ⇝ similar we get $d_N = (1+r_N)\exp\left(-\sigma\sqrt{\frac{T}{N}}\right) - 1$

$\Rightarrow p_N^* = \frac{(1+r_N)\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - 1 - r_N}{(1+r_N)\left(\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)\right)} = \frac{\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - 1}{\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)}$ ⇝ $(1-p_N^*) = \frac{1 - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)}{\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)}$

⇝ with L'Hôspital we have: $\lim_{N\to\infty} N\cdot\mu_N = \lim_{N\to\infty}\frac{-\sigma\sqrt{TN}\exp\left(\sigma\sqrt{\frac{T}{N}}\right) + \sigma\sqrt{TN}}{\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)} + \frac{\sigma\sqrt{TN} - \sigma\sqrt{TN}\exp\left(-\sigma\sqrt{\frac{T}{N}}\right)}{\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)} = \lim_{N\to\infty}\frac{2\sigma\sqrt{TN} - \sigma\sqrt{TN}\left(\exp\left(\sigma\sqrt{\frac{T}{N}}\right) + \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)\right)}{\exp\left(\sigma\sqrt{\frac{T}{N}}\right) - \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)}$

$h = \frac{1}{\sqrt{N}}$, $\eta = \sigma\sqrt{T}$

$= \lim_{h\downarrow 0}\frac{2\eta h^{-1} - \eta h^{-1}(\exp(\eta h) + \exp(-\eta h))}{\exp(\eta h) - \exp(-\eta h)} = \lim_{h\downarrow 0}\frac{2\eta}{h\cdot 2}\cdot\frac{(1-\cosh(\eta h))}{\sinh(\eta h)} \overset{\text{L'Hospital}}{=} \lim_{h\downarrow 0}\left(\frac{-\eta^2\sinh(\eta h)}{\sinh(\eta h) + h\eta\cosh(\eta h)}\right) \overset{\text{L'Hospital}}{=} \lim_{h\downarrow 0}\left(\frac{-\eta^3\cosh(\eta h)}{\eta\cosh(\eta h) + \eta\cosh(\eta h) + h\eta^2\sinh(\eta h)}\right) =$

$= -\frac{\eta^2}{2} = -\frac{\sigma^2 T}{2} =: \mu\in\mathbb{R}$

$\Rightarrow$ the conditions for a) are fulfilled $\overset{a)}{\Rightarrow} Z_N \xrightarrow[N\to\infty]{d} Z\sim N(\mu,\sigma^2 T)$ (3)

⇝ we know from Prop.Th. (Thm 2.7) that (3) implies the existence of R.V. $\tilde{Z}_N$ s.t. $\tilde{Z}_N\sim Z_N$ and s.t. $\tilde{Z}_N \xrightarrow{P\text{-a.s.}} Z$ (4)

$\Rightarrow \lim_{N\to\infty} V_0^{P,N} \overset{(6)}{=} \lim_{N\to\infty} E^*\left(\left(\frac{K}{(1+r_N)^N} - S_0\exp(Z_N)\right)^+\right) \overset{E(\cdot)\text{ depends only on the law of } Z_N}{=} \lim_{N\to\infty} E^*\left(\underbrace{\left(\frac{K}{(1+r_N)^N} - S_0\exp(\tilde{Z}_N)\right)^+}_{=:f(\tilde{Z}_N)}\right) \overset{\text{with (5) \& (4) we can use dom. Conv., since the function } f \text{ is contin.}}{=}$

$= E^*\left(\left(\lim_{N\to\infty}\left(\frac{K}{(1+r_N)^N} - S_0\exp(\tilde{Z}_N)\right)\right)^+\right) = E^*\left(\left(\frac{K}{e^{RT}} - S_0\exp(Z)\right)^+\right) = E^*\left[\left(Ke^{-RT} - S_0\cdot\exp(\mu + \sigma\sqrt{T}W)\right)^+\right] =$

$Z\sim N(\mu,\sigma^2 T) \Leftrightarrow \frac{Z-\mu}{\sigma\sqrt{T}} =: W\sim N(0,1) \Leftrightarrow Z = \sigma\sqrt{T}W + \mu$

$$= \int_{\{w\in\mathbb{R} \mid S_0 \exp(\mu+\sigma\sqrt{T}w)\le K e^{-RT}\} =: D} \left(K e^{-RT} - S_0 \exp(\mu + \sigma\sqrt{T} w)\right) \frac{1}{\sqrt{2\pi}} \exp^{-\frac{w^2}{2}} \, dw =: A + B =: (\mathrm{II})$$

$\rightsquigarrow$ $S_0 \exp(\mu+\sigma\sqrt{T}w) \le K e^{-RT} \overset{\log(\cdot)\text{ increasing}}{\Leftrightarrow} \mu + \sigma\sqrt{T} w \le \log\left(\frac{K e^{-RT}}{S_0}\right) = \log\left(\frac{K}{S_0}\right) - RT \Leftrightarrow w \le \frac{\log(K/S_0) - RT - \mu}{\sigma\sqrt{T}} =: -d_2$

$$\Rightarrow A = \int_D K e^{-RT} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw = K e^{-RT} \int_{-\infty}^{-d_2} \frac{\exp\left(-\frac{w^2}{2}\right)}{\sqrt{2\pi}} dw = K e^{-RT} \Phi(-d_2) \qquad (7)$$

$$\rightsquigarrow B = -\int_D S_0 \exp(\mu+\sigma\sqrt{T}w) \exp\left(-\frac{w^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} dw \overset{\mu = -\frac{\sigma^2 T}{2}}{=} -S_0 \int_{-\infty}^{-d_2} \exp\Big(-\tfrac{1}{2}\underbrace{(w^2 - 2\sigma\sqrt{T}w + \sigma^2 T)}_{=(w-\sigma\sqrt{T})^2 = x^2}\Big) \frac{1}{\sqrt{2\pi}} dw \overset{\text{substitution}}{=} \int_{-\infty}^{-d_2-\sigma\sqrt{T}} -S_0 \cdot \exp\left(-\tfrac{1}{2}x^2\right) \frac{1}{\sqrt{2\pi}} dx =$$

$$= -S_0 \cdot \Phi(\underbrace{-d_2 - \sigma\sqrt{T}}_{=: -d_1}) = -S_0 \cdot \Phi(-d_1) \qquad (8)$$

$$\Rightarrow \lim_{N\to\infty} V_0^{P,N} = (\mathrm{II}) = A + B \overset{(7),(8)}{=} K e^{-RT} \Phi(-d_2) - S_0 \Phi(-d_1), \quad \text{with } d_1 = d_2 + \sigma\sqrt{T} \overset{\mu = -\frac{\sigma^2 T}{2}}{=} \frac{\log(S_0/K) + RT - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} + \sigma\sqrt{T} =$$

$$= \frac{\log(S_0/K) + RT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

$\blacksquare$

Exercise 6.4 Python - Trinomial model Inspire yourself from binomial price array to complete the trinomial price function.

$\rightsquigarrow$ the trinomial model is given by 2 assets, 1 is the numeraire $(B_t)_t$ (e.g.: $B_t = e^{rt}$) and the other is S_t with S_0 fixed and $S_{t+1} = S_t \cdot Y_{t+1}$, with $Y_{t+1} \in \{d, 1, u\}$ a RV.

$\rightsquigarrow$ we choose an (EMM) Q s.t. all Y_t are iid and we need to have that $\frac{S_t}{B_t}$ is a martingale, i.e.:

$$\mathbb{E}_Q\left(\frac{S_t}{B_t} \,\middle|\, \mathcal{F}_{t-1}\right) \overset{!}{=} \frac{S_{t-1}}{B_{t-1}} \Leftrightarrow \frac{S_{t-1}}{B_{t-1}} = \mathbb{E}_Q\left(\frac{S_{t-1} \cdot Y_t}{B_t} \,\middle|\, \mathcal{F}_{t-1}\right) = \frac{1}{B_t} \cdot S_{t-1}\, \mathbb{E}_Q(Y_t) \Leftrightarrow e^r = d\, q_1 + 1\cdot(1-q_1-q_2) + u \cdot q_2$$

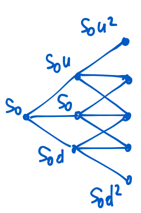
$\rightsquigarrow$ we assume that $d = \frac{1}{u}$ and $q_1, q_2, 1-q_1-q_2 > 0$

$\rightsquigarrow$ obviously Q is not unique

$\rightsquigarrow$ one possibility is to use (see paper on trinomial trees): $u = e^{\sigma\sqrt{2\Delta t}}$, $d = e^{-\sigma\sqrt{2\Delta t}}$, $B_n = e^{r\cdot\Delta t\cdot n}$, $q_u = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2$

$$p_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2, \quad p_m = 1 - p_u - p_d$$

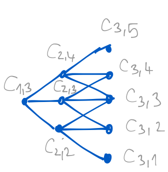
$\rightsquigarrow$ since $d = \frac{1}{u}$ we get 2 new possible values in each step $\rightsquigarrow$ a corresponding tree looks like:



$\rightsquigarrow$ if we have n steps, i.e. S_{n-1} is the value at maturity then we know that S_{n-1} can take $1 + 2(n-1)$ values

$\rightsquigarrow$ we want to find a recursion formula for the prices of an option C

$\rightsquigarrow$ we want to find arbitrage free prices wrt. our fixed (EMM) Q $\Rightarrow$ the discounted price process $\pi_t\left(\frac{C}{B_T}\right)$ is an Q-martingale (otherwise we wouldn't get an arbitrage-free price) $\rightsquigarrow$ writing $(C_{n,j})_j$ for the possible values of the undiscounted price process at time n, we get from the martingale property:



$$\mathbb{E}_Q\left(\frac{C_{n+1}}{B_{n+1}} \,\middle|\, C_{n,j}\right) = \frac{C_{n,j}}{B_n} \Leftrightarrow \frac{B_n}{B_{n+1}} \cdot \underbrace{\mathbb{E}_Q(C_{n+1} \mid C_{n,j})}_{= q_u C_{n+1,j+1} + q_d C_{n+1,j-1} + (1-q_u-q_d) C_{n+1,j}} = C_{n,j}$$

$\Rightarrow$ this yields the recursion formula: $C_{n,j} = \frac{B_n}{B_{n+1}} \cdot \left(q_u C_{n+1,j+1} + q_d C_{n+1,j-1} + q_m C_{n+1,j}\right)$

$\rightsquigarrow$ use an algorithm that computes everything (the $C_{n,j}$ values) only in one vector (not a matrix)