Introduction to Mathematical Finance

Solution sheet 6

Solution 6.1

(a) If $(N_t)_{t\in\{0,..,T\}}$ a *numeraire*, $\left(\frac{S_t}{B_t}\right)_t$ is a Q-EMM $\iff$ $\left(\frac{S_t}{N_t}\right)_t$ is a Q^N-EMM where

$$\left.\frac{dQ^N}{dQ}\right|_{\mathcal{F}_t} = \frac{N_T/N_t}{B_T/B_t}.$$

(b) The forward Libor rate $(L(t,T,T+\delta))_t$ is a martingale under the *forward probability.*

$$\begin{aligned} L(t,T,T+\delta) =& \frac{1}{\delta}\left(\frac{1}{P(t,T,T+\delta)} - 1\right) \\ =& \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right) \end{aligned}$$

By the numeraire change theorem, if we take as numeraire the zero-coupon bond $(N_t)_{t\in\{0,..,T\}} = (P(t,T+\delta))_{t\in\{0,..,T\}}$ then we have that

$$\begin{aligned} \left(\frac{P(t,T)}{B_t}\right)_t \text{ is a } Q\text{-EMM} &\iff \left(\frac{P(t,T)}{P(t,T+\delta)}\right)_t \text{ is a } Q^{T+\delta}\text{-EMM} \\ &\iff \left(\frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)\right)_t \text{ is a } Q^{T+\delta}\text{-EMM} \\ &\iff (L(t,T,T+\delta))_t \text{ is a } Q^{T+\delta}\text{-EMM}. \end{aligned}$$

The arbitrage free price of a caplet with payoff $\mathrm{C}^{caplet} = \delta(L(t,T,T+\delta) - K)_+$ is

$$\begin{aligned} \pi(\mathrm{C}^{caplet}) =& E_t^Q\left(\frac{\mathrm{C}^{caplet}}{B_{T+\delta}/B_t}\right) \\ =& B_t\ E_t^Q\left(\frac{\delta(L(T,T,T+\delta)-K)_+}{B_{T+\delta}}\right) \\ =& B_t\ E_t^{Q^N}\left(\left.\frac{dQ}{dQ^N}\right|_{\mathcal{F}_t}\frac{\delta(L(T,T,T+\delta)-K)_+}{B_{T+\delta}}\right) \\ =& B_t\ E_t^{Q^N}\left(\frac{B_{T+\delta}/B_t}{N_{T+\delta}/N_t}\frac{\delta(L(T,T,T+\delta)-K)_+}{B_{T+\delta}}\right) \\ =& N_t\ E_t^{Q^N}\left(\frac{\delta(L(T,T,T+\delta)-K)_+}{N_{T+\delta}}\right). \end{aligned}$$

The numeraire here is the *zero-coupon* $N_t = P(t,T+\delta)$.

$$\begin{aligned} \pi(\mathrm{C}^{caplet}) =& P(t,T+\delta)\ E_t^{Q^{T+\delta}}\left(\frac{\delta(L(T,T,T+\delta)-K)_+}{P(T+\delta,T+\delta)}\right) \\ =& P(t,T+\delta)\ E_t^{Q^{T+\delta}}\left(\delta(L(T,T,T+\delta)-K)_+\right) \end{aligned}$$

(c) The swap rate $(S_{T_0,T_N}(t))_t$ is a martingale under the *forward probability*.

$$\left(\frac{P(t,T_i)}{B_t}\right)_t \text{ is a } Q\text{-EMM } (i=0,N) \iff \left(\frac{P(t,T_i)}{A(t)}\right)_t \text{ is a } Q^A\text{-EMM } (i=0,N)$$
$$\iff \left(\frac{P(t,T_0)-P(t,T_N)}{A(t)}\right)_t \text{ is a } Q^A\text{-EMM}$$
$$\iff (S_{T_0,T_N}(t))_t \text{ is a } Q^A\text{-EMM}$$

The arbitrage free price of a swaption with payoff $\mathrm{C}^{swaption} = A(T_f)(S_{T_0,T_N}(t)-K)_+$ is

$$\begin{aligned}\pi(\mathrm{C}^{swaption}) =& E_t^Q\left(\frac{\mathrm{C}^{swaption}}{B_T/B_t}\right)\\ =& B_t\ E_t^Q\left(\frac{A(T_f)(S_{T_0,T_N}(T_f)-K)_+}{B_T}\right)\\ =& A(t)\ E_t^{Q^A}\left(\frac{A(T_f)(S_{T_0,T_N}(T_f)-K)_+}{A(T_f)}\right)\\ =& A(t)\ E_t^{Q^A}\left((S_{T_0,T_N}(T_f)-K)_+\right)\end{aligned}$$

Solution 6.2

(a) Book chapter 5.5.2

(b) Book chapter 5.6.1

(c) Book chapter 5.53

(d) Book chapter 5.62

Solution 6.3

(a) Let us compute the characteristic function of Z_n :

$$\begin{aligned}\phi_{Z_n}(z) &= \mathbb{E}\left[e^{izZ_n}\right]\\ &= \mathbb{E}\left[e^{iz\sum_{i=1}^n X_i^n}\right]\\ &= \prod_{i=1}^n \mathbb{E}\left[e^{izX_i^n}\right]\\ &= \mathbb{E}\left[e^{izX_1^n}\right]^n\end{aligned}$$

where we get the third equality by independence of the X_i^n, and the fourth because they are identically distributed. A Taylor expansion gives :

$$\mathbb{E}\left[e^{izX_1^n}\right] = \left(1 + iz\mu_n - \frac{z^2\sigma^2 T}{2n} + o\left(\frac{1}{n}\right)\right).$$

Hence $\lim_{n\to\infty}\phi_{Y_n}(z) = \exp\left(iz\mu - \frac{\sigma^2 z^2 T}{2}\right)$, which is continuous at 0 and the characteristic function of a normal random variable with mean μ and variance $\sigma^2 T$.

(b) The price at time 0 of an attainable claim is given by the expected value under an equivalent martingale measure of its discounted payoff :

$$\begin{aligned} V_0^{P,N} &= \frac{\mathbb{E}^*\left[\left(K - \widetilde{S}_N^1\right)^+\right]}{(1+r)^N} \\ &= \frac{\mathbb{E}^*\left[\left(K - S_0^1 \prod_{i=1}^N Y_i\right)^+\right]}{(1+r)^N} \\ &= \mathbb{E}^*\left[\left(\frac{K}{(1+r)^N} - S_0^1 \exp\left(Z_N\right)\right)^+\right], \end{aligned}$$

where $Z_N = \sum_{i=1}^N \log\left(\frac{Y_i}{1+r}\right)$.

(c) By assumption, $\log\left(\frac{Y_i}{1+r}\right)$ is valued in $\left\{-\sigma\sqrt{\frac{T}{N}},\ \sigma\sqrt{\frac{T}{N}}\right\}$ and the Y_i's are i.i.d. under $\mathbb{P}^*$. Furthermore we have :

$$\begin{aligned} \mathbb{E}^*\left[\log\left(\frac{Y_i}{1+r}\right)\right] &= -p^*\sigma\sqrt{\frac{T}{N}} + (1-p^*)\sigma\sqrt{\frac{T}{N}} \\ &= (1-2p^*)\sigma\sqrt{\frac{T}{N}} \\ &= \left(1 - 2\frac{u-r}{u-d}\right)\sigma\sqrt{\frac{T}{N}} \\ &= \left(1 - 2\frac{e^{\sigma\sqrt{\frac{T}{N}}}\left(1+\frac{RT}{N}\right) - 1 - \frac{RT}{N}}{e^{\sigma\sqrt{\frac{T}{N}}}\left(1+\frac{RT}{N}\right) - e^{-\sigma\sqrt{\frac{T}{N}}}\left(1+\frac{RT}{N}\right)}\right)\sigma\sqrt{\frac{T}{N}} \\ &= \frac{2 - e^{\sigma\sqrt{\frac{T}{N}}} - e^{-\sigma\sqrt{\frac{T}{N}}}}{e^{\sigma\sqrt{\frac{T}{N}}} - e^{-\sigma\sqrt{\frac{T}{N}}}}\sigma\sqrt{\frac{T}{N}} \\ &= \frac{2 - \left(1+\sigma\sqrt{\frac{T}{N}} + \frac{\sigma^2 T}{2N} + o\left(\frac{1}{N}\right)\right) - \left(1-\sigma\sqrt{\frac{T}{N}} + \frac{\sigma^2 T}{2N} + o\left(\frac{1}{N}\right)\right)}{\left(1+\sigma\sqrt{\frac{T}{N}} + \frac{\sigma^2 T}{2N} + o\left(\frac{1}{N}\right)\right) - \left(1-\sigma\sqrt{\frac{T}{N}} + \frac{\sigma^2 T}{2N} + o\left(\frac{1}{N}\right)\right)}\sigma\sqrt{\frac{T}{N}} \\ &= \frac{-\frac{\sigma^2 T}{N} + o\left(\frac{1}{N}\right)}{2\sigma\sqrt{\frac{T}{N}} + o\left(\frac{1}{N}\right)}\sigma\sqrt{\frac{T}{N}} \\ &= \frac{-\frac{\sigma^2 T}{N} + o\left(\frac{1}{N}\right)}{2 + o\left(\sqrt{\frac{1}{N}}\right)} \\ &= -\frac{\sigma^2 T}{2N} + o\left(\frac{1}{N}\right) \end{aligned}$$

and we can use the result proved in a) with $\mu = -\frac{\sigma^2 T}{2}$.

Consider the function $f(z) = \left(Ke^{-RT} - S_0 e^y\right)^+$ and the difference :

$$\begin{aligned} \left|V_0^{P,N} - \mathbb{E}^*\left[f(Z_N)\right]\right| &= \left|\mathbb{E}^*\left[\left(\frac{K}{(1+r)^N} - S_0^1\exp\left(Z_N\right)\right)^+\right] - \mathbb{E}^*\left[\left(Ke^{-RT} - S_0^1\exp\left(Z_N\right)\right)^+\right]\right| \\ &\leqslant K\left|\left(1+\frac{RT}{N}\right)^{-N} - e^{-RT}\right| \end{aligned}$$

Hence, we have $\lim_{N\to\infty} V_0^{P,N} = \lim_{N\to\infty} \mathbb{E}^*\left[f(Z_N)\right]$. And since f is bounded and continuous, and the sequence $(Z_N)_{N\in\mathbb{N}}$ converges in law to a normal random variable with mean $\mu = -\frac{\sigma^2 T}{2}$ and variance $\sigma^2 T$, we have :

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}^*\left[f(Z_N)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(Ke^{-RT} - S_0 e^{\sigma\sqrt{T}y - \frac{\sigma^2 T}{2}}\right)^+ e^{-\frac{y^2}{2}} dy \\
&= Ke^{-RT} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{\{Ke^{-RT} \geqslant S_0 e^{\sigma\sqrt{T}y - \frac{\sigma^2 T}{2}}\}} e^{-\frac{y^2}{2}} dy \\
&\quad - S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{\{Ke^{-RT} \geqslant S_0 e^{\sigma\sqrt{T}y - \frac{\sigma^2 T}{2}}\}} e^{\sigma\sqrt{T}y - \frac{\sigma^2 T}{2}} e^{-\frac{y^2}{2}} dy \\
&= Ke^{-RT} \Phi\left(-d_2\right) - S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-\frac{1}{2}\left(y - \sigma\sqrt{T}\right)^2} dy \\
&= Ke^{-RT} \Phi\left(-d_2\right) - S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_1} e^{-\frac{1}{2}y^2} dy \\
&= Ke^{-RT} \Phi\left(-d_2\right) - S_0 \Phi\left(-d_1\right)
\end{aligned}$$

Which gives the result.

Solution 6.4 The solution will be given the next week.