Introduction to Mathematical Finance

Exercise sheet 7

Exercise 7.1 Let $(\widetilde{S}^0, \widetilde{S}^1)$ be a binomial model with $\widetilde{S}^1_0 := 1$ and $u > r > d > -1$. Denote by $(\widehat{S}^0, \widehat{S}^1)$ the market discounted with $\widetilde{S}^1$, i.e.

$$\widehat{S}^0 := \frac{\widetilde{S}^0}{\widetilde{S}^1} \quad \text{and} \quad \widehat{S}^1 := \frac{\widetilde{S}^1}{\widetilde{S}^1} \equiv 1\,.$$

(a) Show that there exists a unique equivalent martingale measure Q^{**} for $\widehat{S}^0$.

(b) Let Q^* be the unique equivalent martingale measure for S^1. Show that the density of Q^{**} with respect to Q^* on $\mathcal{F}_T$ is given by

$$\frac{\mathrm{d}Q^{**}}{\mathrm{d}Q^*} = S^1_T\,.$$

(c) Show that for an *undiscounted* payoff $\widetilde{H} \in L^0_+(\mathcal{F}_T)$ we have

$$\widetilde{S}^0_k\, E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\middle|\mathcal{F}_k\right] = \widetilde{S}^1_k\, E_{Q^{**}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\middle|\mathcal{F}_k\right], \quad k = 0, \dots, T\,.$$

This formula shows that the risk-neutral pricing method is invariant under the so-called *change of numéraire*.

Exercise 7.2 Consider the trinomial model with $r = 0.05$ and $T = 1$. Suppose that the evolution of $\left(\widetilde{S}^0, \widetilde{S}^1\right)$ is given by

$$\widetilde{S}^1_0 = S^1_0 = s_0 = 80, \quad \widetilde{S}^1_1 = \begin{cases} 120 & \text{with probability } 0.2 \\ 90 & \phantom{\text{with probability }} 0.3 \\ 60 & \phantom{\text{with probability }} 0.5 \end{cases}, \text{ and } \widetilde{S}^0_k = (1+r)^k\,, \text{ for } k \in \{0, 1\}.$$

(a) Compute the set of all arbitrage-free prices for the European call option $\widetilde{H} = (\widetilde{S}^1_1 - 80)^+$.

(b) Find the set of all attainable contingent claims.

(c) Is it possible to replicate the previous call option by a self-financing portfolio?

Exercise 7.3 Let X be any adapted, integrable process.

(a) Show that there exist a martingale $M = (M_k)_{k\in\mathbb{N}_0}$ and a predictable process $A = (A_k)_{k\in\mathbb{N}}$ with $A_0 := 0$ and

$$X_k = X_0 + M_k + A_k \quad P\text{-a.s.},$$

for $k \in \mathbb{N}_0$ and $M_0 = A_0 = 0$ P-a.s.

(b) Show that M and A are unique up to identification P-a.s.

(c) Show that X is a supermartingale if and only if A is decreasing, i.e., $A_{k+1} \leq A_k$ P-a.s. for all $k \in \mathbb{N}_0$. (So then we can write $X = X_0 + M - B$ with an increasing predictable process B null at 0.)

Exercise 7.4 Python - Trinomial model Inspire yourself from binomial price array to complete the trinomial price function.

```
def trinomial_price(maturity, spot, strike, rate, vol, steps_number, payoff_fct=None, graph_name=None):
  """Compute the trinomial price. Draw graph if graph_name is given.
  """
  """Compute the trinomial price. Draw graph if graph_name is given.
  """
  deltaT = maturity / steps_number
  discount_factor = exp(-rate * deltaT)
  up = exp(vol * sqrt(2*deltaT))
  down = 1 / up
  denominator = exp(vol * sqrt(deltaT/2)) - exp(-vol * sqrt(deltaT/2))
  proba_up = ((exp(rate * deltaT/2) - exp(-vol * sqrt(deltaT/2))) / denominator) ** 2
  proba_down = ((exp(vol * sqrt(deltaT/2))- exp(rate * deltaT/2)) / denominator) ** 2
  proba_middle = 1 - proba_up - proba_down
  steps = range(steps_number)
  spot_prices = [spot * up ** i for i in reversed(steps[1:])] + [spot] + [spot * down ** i for i in steps[1:]]
  option_prices = [payoff_fct(spot_price, strike) for spot_price in spot_prices]


  # The following two list are only needed to display the graph:
  spot_prices_history = [spot_prices]
  option_prices_history = [option_prices]

  # TODO

  if graph_name:
    create_graph(graph_name, spot_prices_history, option_prices_history)

  return option_prices[0]
```