Introduction to Mathematical Finance

Solution sheet 7

Solution 7.1 Assume without loss of generality that $\Omega = \{1,2\}^T$ and $Y_k((x_1,\dots,x_T)) = 1 + y_{x_k}$, where $y_1 := d$ and $y_2 := u$.

(a) **First method.** Any measure $Q \approx P$ on $\mathcal{F}_T$ can be described by its transition probabilities q_{x_1}, q_{x_1,x_2}, …, $q_{x_1,\dots,x_T}$, where x_1, …, $x_k \in \{1,2\}$ and

$$\begin{aligned} q_{x_1} &:= Q[Y_1 = 1 + y_{x_1}], \\ q_{x_1,\dots,x_k} &:= Q[Y_k = 1 + y_{x_k} \,|\, Y_1 = 1 + y_{x_1}, \dots, Y_{k-1} = 1 + y_{x_{k-1}}], \quad k = 2,\dots,T. \end{aligned} \tag{1}$$

Since $\widehat{S}^1$ is finitely valued (and hence bounded) and adapted, it is a Q-martingale if and only if for all $k = 0,\dots,T-1$, we have

$$E_Q\left[\widehat{S}^0_{k+1} \,\middle|\, \mathcal{F}_k\right] = \widehat{S}^0_k \quad Q\text{-a.s.}\,. \tag{2}$$

By definition of $\widehat{S}^0$ and since it is strictly positive, the martingale property (2) holds if and only if

$$E_Q\left[\widehat{S}^0_k \frac{1+r}{Y_{k+1}} \,\middle|\, \mathcal{F}_k\right] = \widehat{S}^0_k \quad Q\text{-a.s.}\,,$$

which holds if and only if

$$E_Q\left[\frac{1+r}{Y_{k+1}} \,\middle|\, \mathcal{F}_k\right] = 1 \quad Q\text{-a.s.}\,.$$

Note that we do not know a priori whether the Y_k are independent under Q. Since $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_k = \sigma(Y_1,\dots,Y_k)$ for $k = 1,\dots T$, and Y_k only takes two values, $\widehat{S}^0$ is an Q-martingale if and only if $E_Q\left[\frac{1+r}{Y_1}\right] = 1$ and for all $k \in \{1,\dots,T-1\}$ and all x_1, …, $x_k \in \{1,2\}$ we have

$$E_Q\left[\frac{1+r}{Y_{k+1}} \,\middle|\, Y_1 = 1 + y_{x_1}, \dots, Y_k = 1 + y_{x_k}\right] = 1\,. \tag{3}$$

Now, we have

$$\begin{aligned} E_Q\left[\frac{1+r}{Y_1}\right] = 1 \quad &\Leftrightarrow \quad \frac{1+r}{1+d} q_1 + \frac{1+r}{1+u}(1-q_1) = 1 \\ &\Leftrightarrow \quad \left(\frac{1+u}{1+d} - 1\right) q_1 = \frac{1+u}{1+r} - 1 \\ &\Leftrightarrow \quad \frac{u-d}{1+d} q_1 = \frac{u-r}{1+r} \\ &\Leftrightarrow \quad q_1 = \frac{1+d}{1+r}\frac{u-r}{u-d}\,. \end{aligned} \tag{4}$$

Similarly, for all $k \in \{1,\dots,T-1\}$ and all x_1, …, $x_k \in \{1,2\}$, we have

$$E_Q\left[\frac{1+r}{Y_{k+1}} \,\middle|\, Y_1 = 1 + y_{x_1}, \dots, Y_k = 1 + y_{x_k}\right] = 1 \Leftrightarrow \quad q_{x_1,\dots,x_k,1} = \frac{1+d}{1+r}\frac{u-r}{u-d}\,.$$

Note that $q_{x_1,\dots,x_k,1}$ does not depend on $x_1, \dots, x_k$ and equals q_1. Hence, we may conclude that there exists a unique equivalent martingale measure Q^{**} for $\widehat{S}^0$, under which $Y_1, \dots, Y_k$ are i.i.d. and we have

$$Q^{**}[Y = 1+d] = \frac{1+d}{1+r}\frac{u-r}{u-d} =: q_1^{**} \quad \text{and} \quad Q^{**}[Y = 1+u] = \frac{1+u}{1+r}\frac{r-d}{u-d} =: q_2^{**} .$$

Second method: For $k = 1, \dots, T$, set $\widehat{Y}_k = \frac{1+r}{Y_k}$. Then $\widehat{S}^1_k = 1^k = 1$ and $\widehat{S}^0_k = \prod_{j=1}^k \widehat{Y}_j$ for $k = 0, \dots, T$, where the $\widehat{Y}_k$ are independent under P and take the two values $1+\hat{u}$ and $1+\hat{d}$ with probability p_1 and p_2, respectively, where $\hat{u} = \frac{r-d}{1+d} > 0$ and $\hat{d} = \frac{r-u}{1+u} < 0$. In conclusion, $(\widehat{S}^0, \widehat{S}^0)$ can be viewed as a binomial model with $\hat{u} > \hat{r} = 0 > \hat{d}$.

Variant (i): Recalling that $\hat{u}$ corresponds to p_1 and $\hat{d}$ to p_2, it follows that the unique equivalent martingale measure Q^{**} for $\widehat{S}^1$ is given by

$$Q^{**}\left[\{(x_1, \dots, x_T\}\right] := \prod_{k=1}^T q^{**}_{x_k}, \quad x_1, \dots, x_T \in \{1,2\}, \tag{5}$$

where

$$\begin{aligned} q_1^{**} &= \frac{\hat{r}-\hat{d}}{\hat{u}-\hat{d}} = \frac{\frac{u-r}{1+u}}{\frac{r-d}{1+d} - \frac{r-u}{1+u}} = \frac{1+d}{1+r}\frac{u-r}{u-d} \\ q_2^{**} &= \frac{\hat{u}-\hat{r}}{\hat{u}-\hat{d}} = \frac{\frac{r-d}{1+d}}{\frac{r-d}{1+d} - \frac{r-u}{1+u}} = \frac{1+u}{1+r}\frac{r-d}{u-d} . \end{aligned} \tag{6}$$

Variant (ii): We are looking for the unique strictly positive Q^*-martingale $Z^{Q^{**};Q^*}$ starting at 1 such that

$$\frac{\widetilde{S}^0}{\widetilde{S}^1} Z^{Q^{**};Q^*} \text{ is a } Q^*\text{-martingale.}$$

But we already know that $\frac{\widetilde{S}^1}{\widetilde{S}^0}$ is a Q^*-martingale and strictly positive. Hence by uniqueness,

$$Z^{Q^{**};Q^*} = \frac{\widetilde{S}^1}{\widetilde{S}^0} = S^1 .$$

(b) The unique equivalent martingale measure Q^* for $S^1 = \frac{\widetilde{S}^1}{\widetilde{S}^0}$ on $\mathcal{F}_T$ is given by

$$Q^*\left[\{(x_1, \dots, x_T)\}\right] := \prod_{j=1}^T q^*_{x_j} , \tag{7}$$

where $q_1^* = \frac{u-r}{u-d}$ and $q_2^* = \frac{r-d}{u-d}$. By part (a), we have

$$\frac{q_1^{**}}{q_1^*} = \frac{1+d}{1+r} \quad \text{and} \quad \frac{q_2^{**}}{q_2^*} = \frac{1+u}{1+r} . \tag{8}$$

Thus, since Ω is finite, we may deduce that for all $(x_1, \dots, x_T) \in \{1,2\}^T$ we have

$$\begin{aligned} \frac{\mathrm{d}Q^{**}}{\mathrm{d}Q^*}((x_1, \dots, x_T)) &= \frac{Q^{**}\left[\{(x_1, \dots, x_T)\}\right]}{Q^*\left[\{(x_1, \dots, x_T)\}\right]} = \prod_{k=1}^T \frac{q^{**}_{x_k}}{q^*_{x_k}} \\ &= \frac{\prod_{k=1}^T (1+y_{x_k})}{(1+r)^T} = \frac{\prod_{k=1}^T Y_k((x_1, \dots, x_T))}{(1+r)^k} \\ &= \frac{\widetilde{S}^1_T((x_1, \dots, x_T))}{\widetilde{S}^0_T((x_1, \dots, x_T))} . \end{aligned} \tag{9}$$

(c) Denote by Z the density process of Q^{**} with respect to Q^*. Let $k \in \{1, \ldots, T\}$ and $\widetilde{H} \in L^0_+(\mathcal{F}_T)$. Since $Z_T = \frac{\widetilde{S}^1_T}{\widetilde{S}^0_T} = S^1_T$ by part (b) and S^1 is a Q^*-martingale, it follows by the definition of a density process that for $k = 0, \ldots, T$ we have

$$Z_k = E_{Q^*}\left[Z_T \,|\, \mathcal{F}_k\right] = E_{Q^*}\left[S^1_T \,\middle|\, \mathcal{F}_k\right] = S^1_k = \frac{\widetilde{S}^1_k}{\widetilde{S}^0_k} \quad Q^*\text{-a.s.} \ . \tag{10}$$

Thus, by the Bayes formula we get for $k = 0, \ldots, T$

$$\widetilde{S}^1_k \, E_{Q^{**}}\left[\frac{\widetilde{H}}{\widetilde{S}^1_T}\,\middle|\, \mathcal{F}_k\right] = \frac{\widetilde{S}^1_k}{Z_k} E_{Q^*}\left[\frac{Z_T \widetilde{H}}{\widetilde{S}^1_T}\,\middle|\, \mathcal{F}_k\right] = \widetilde{S}^0_k \, E_{Q^*}\left[\frac{\widetilde{H}}{\widetilde{S}^0_T}\,\middle|\, \mathcal{F}_k\right]. \tag{11}$$

Solution 7.2 We use the following notations:

$$S^1_0 = 80, \qquad \tilde{S}^1_1 = \begin{cases} 80(1+y_1) & \text{with probability } p_1 = 0.2 \\ 80(1+y_2) & \qquad\qquad\qquad p_2 = 0.3 \\ 80(1+y_3) & \qquad\qquad\qquad p_3 = 0.5 \end{cases},$$

with $y_1 = \frac{1}{2}, y_2 = \frac{1}{8}, y_3 = -\frac{1}{4}$. Let $q_i = \mathbb{Q}\left[\{\tilde{S}^1_1 = s_0(1+y_i)\}\right]$.

(a) We work on the (finite) path space Ω and use the filtration generated by the price process. S^1 is therefore adapted and integrable, it is then a martingale under $\mathbb{Q}$ if and only if $\mathbb{E}_{\mathbb{Q}}[S^1_1] = S^1_0$.

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[S^1_1] = S^1_0 \quad &\Leftrightarrow \quad \mathbb{E}_{\mathbb{Q}}[\tilde{S}^1_1] = S^1_0(1+r) \\ \mathbb{Q} \text{ is an EMM} \quad &\Leftrightarrow \quad \begin{cases} 120q_1 + 90q_2 + 60q_3 = 80 \cdot 1.05 \\ q_1 + q_2 + q_3 = 1 \\ 0 < q_1, q_2, q_3 < 1 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} q_2 = 0.8 - 2q_1 \\ q_3 = 0.2 + q_1 \\ q_1 \in (0, 0.4) \end{cases} \end{aligned}$$

The set of all equivalent martingale measures is given by

$$P_e(S) = \{\mathbb{Q}_\lambda = (\lambda, 0.8 - 2\lambda, 0.2 + \lambda) \mid \lambda \in (0, 0.4)\}.$$

The set of all arbitrage-free prices is given by (we can use the risk-neutral valuation formula)

$$\begin{aligned} C &= \{c_\lambda := \mathbb{E}_{\mathbb{Q}_\lambda}\Big[\frac{(\tilde{S}^1_1 - 80)^+}{1+r}\Big] \mid \lambda \in (0, 0.4)\} \\ &= \{c_\lambda = \frac{1}{1.05}\Big(40 \cdot \lambda + 10(0.8 - 2\lambda)\Big) \mid \lambda \in (0, 0.4)\} \\ &= \{c_\lambda = \frac{1}{1.05}(20\lambda + 8) \mid \lambda \in (0, 0.4)\}. \end{aligned}$$

This set is the whole open interval $(7.619, 15.238)$.

(b) $\tilde{H} \in L^0_+$ can be replicated if there exists an admissible self-financing strategy $\varphi = (\varphi^0, \vartheta)$ such that

$$\varphi^0_1(1+r) + \vartheta_1 \tilde{S}^1_1 = \tilde{H}.$$

Let $\tilde{H}^{y_i}$ be the value of the payoff if $\tilde{S}^1_1 = s_0(1+y_i)$, we are looking for non trivial solutions of the following system:

$$\begin{bmatrix} 1.05 & 120 \\ 1.05 & 90 \\ 1.05 & 60 \end{bmatrix} \cdot \begin{bmatrix} \varphi^0_1 \\ \vartheta_1 \end{bmatrix} = \begin{bmatrix} \tilde{H}^{y_1} \\ \tilde{H}^{y_2} \\ \tilde{H}^{y_3} \end{bmatrix}.$$

This system admits non trivial solutions if and only if

$$\det \begin{bmatrix} 1.05 & 120 & \tilde{H}^{y_1} \\ 1.05 & 90 & \tilde{H}^{y_2} \\ 1.05 & 60 & \tilde{H}^{y_3} \end{bmatrix} = 0$$

In order to be attainable a contingent claim must satisfy

$$-\tilde{H}^{y_1} + 2\tilde{H}^{y_2} - \tilde{H}^{y_3} = 0$$

(c) It's easy to check that our call option doesn't satisfy the previous equation.

Solution 7.3

(a) Define the processes by

$$M_k = \sum_{j=1}^{k} (X_j - E[X_j|\mathcal{F}_{j-1}])$$

and

$$A_k = \sum_{j=1}^{k} (E[X_j|\mathcal{F}_{j-1}] - X_{j-1}).$$

These processes are well-defined since X is integrable. Note that both processes are integrable, M is adapted, and A is predictable. Furthermore,

$$E[\Delta M_{k+1}|\mathcal{F}_k] = E[X_{k+1} - E[X_{k+1}|\mathcal{F}_k]|\mathcal{F}_k] = 0 \quad P\text{-a.s.}$$

Hence, M is a martingale.

(b) Suppose that also M' and A' satisfy $X = X_0 + M' + A' = X_0 + M + A$. Then $M' - M = A - A'$. Denote this process by Y and observe that it is both a martingale and predictable. Thus,

$$Y_{k-1} = E[Y_k|\mathcal{F}_{k-1}] = Y_k \quad P\text{-a.s.},$$

hence Y is constant. Since $M_0 = M_0' = 0$ P-a.s., $Y = 0$ P-a.s., showing that the decomposition is almost surely unique.

(c) Suppose X is a supermartingale. Then, for every $k \in \mathbb{N}_0$,

$$0 \geq E[\Delta X_{k+1}|\mathcal{F}_k] = E[\Delta M_{k+1} + \Delta A_{k+1}|\mathcal{F}_k] = \Delta A_{k+1} \quad P\text{-a.s.},$$

where we use that M is a martingale and A is predictable. This shows that $A_{k+1} \leq A_k$ P-a.s. For the converse, suppose $\Delta A_{k+1} \leq 0$ P-a.s. Then,

$$E[\Delta X_{k+1}|\mathcal{F}_k] = E[\Delta M_{k+1} + \Delta A_{k+1}|\mathcal{F}_k] \leq E[\Delta M_{k+1}|\mathcal{F}_k] = 0 \quad P\text{-a.s.}$$

for all $k \in \mathbb{N}_0$. We thus conclude that X is a supermartingale.

Solution 7.4

```
def trinomial_price(maturity, spot, strike, rate, vol, steps_number, payoff_fct
    =None, graph_name=None):
  """Compute the trinomial price. Draw graph if graph_name is given.
  """
  deltaT = maturity / steps_number
  discount_factor = exp(-rate * deltaT)
```

```python
up = exp(vol * sqrt(2*deltaT))
down = 1 / up
denominator = exp(vol * sqrt(deltaT/2)) - exp(-vol * sqrt(deltaT/2))
proba_up = ((exp(rate * deltaT/2) - exp(-vol * sqrt(deltaT/2))) / denominator) ** 2
proba_down = ((exp(vol * sqrt(deltaT/2))- exp(rate * deltaT/2)) / denominator) ** 2
proba_middle = 1 - proba_up - proba_down
steps = range(steps_number)
spot_prices = [spot * up ** i for i in reversed(steps[1:])] + [spot] + [spot * down ** i for i in steps[1:]]
option_prices = [payoff_fct(spot_price, strike) for spot_price in spot_prices]

# The following two list are only needed to display the graph:
spot_prices_history = [spot_prices]
option_prices_history = [option_prices]

def next_option_price(spot_price, price_up, price_midlle, price_down):
  option_price = (discount_factor *(proba_up * price_up +
                               proba_middle * price_midlle +
                               proba_down * price_down))

while len(option_prices) > 1:
  option_prices = [next_option_price(spot_prices[i], *option_prices[i-1:i+2])
                   for i in range(1, len(option_prices)-1)]
  spot_prices = spot_prices[1:-1]
  spot_prices_history.insert(0, spot_prices)
  option_prices_history.insert(0, option_prices)

if graph_name:
  create_graph(graph_name, spot_prices_history, option_prices_history)

return option_prices[0]
```