1. a) We know by completeness of the binomial market that there exists a unique EMM Q^* for the discounted market $S^0 = \frac{\tilde S^0}{\tilde S^0} \equiv 1$ and $S^1 = \frac{\tilde S^1}{\tilde S_0}$.

By the change of numéraire theorem we know that $M_e(\tilde S^1) = \{Q^{**} \mid E_Q[\frac{dQ^{**}}{dQ} \mid \mathcal{F}_t] = \frac{\tilde S^0_0}{\tilde S^1_0} \frac{\tilde S^1_t}{\tilde S^0_t}$ for some $Q \in M_e(\tilde S^0)\}$. But since $M_e(\tilde S^0) = \{Q^*\} \Rightarrow \# M_e(\tilde S^1) = 1$ so there is only 1 EMM Q^{**} for $\hat S^0$ and we have $\frac{dQ^{**}}{dQ^*} = \frac{\tilde S^0_0}{\tilde S^1_0} \cdot \frac{\tilde S^1_T}{\tilde S^0_T}$

b) We know that $\frac{dQ^{**}}{dQ^*} = \frac{\tilde S^0_0}{\tilde S^1_0} \cdot \frac{\tilde S^1_T}{\tilde S^0_T} = {S^1_0}^{-1} \cdot S^1_T = S^1_T$ since $S^1_0 = \tilde S^1_0 / (1+r)^0 = 1$

c)
$$\tilde S^1_k E_{Q^{**}}[\frac{\tilde H}{\tilde S^1_T} \mid \mathcal{F}_k] = E_{Q^{**}}[\frac{\tilde H}{\tilde S^1_T} \tilde S^1_k \mid \mathcal{F}_k] = E_{Q^*}[\frac{dQ^{**}}{dQ^*} \mid \mathcal{F}_k]^{-1} E_{Q^*}[\frac{\tilde H}{\tilde S^1_T} \tilde S^1_k E_{Q^*}[\frac{dQ^{**}}{dQ^*} \mid \mathcal{F}_T] \mid \mathcal{F}_k]$$
$$= E_{Q^*}[S^1_T \mid \mathcal{F}_k]^{-1} \cdot E_{Q^*}[\frac{\tilde H}{\tilde S^1_T} \tilde S^1_k E_{Q^*}[S^1_T \mid \mathcal{F}_T] \mid \mathcal{F}_k]$$
$$= (S^1_k)^{-1} \cdot E_{Q^*}[\frac{\tilde H}{\tilde S^1_T} \tilde S^1_k S^1_T \mid \mathcal{F}_k] = \frac{\tilde S^0_k}{\tilde S^1_k} \cdot \tilde S^1_k \cdot E_{Q^*}[\frac{\tilde H}{\tilde S^1_T} \cdot \frac{\tilde S^1_T}{\tilde S^0_T} \mid \mathcal{F}_k]$$
$$= \tilde S^0_k \cdot E_{Q^*}[\frac{\tilde H}{\tilde S^0_T} \mid \mathcal{F}_k]$$

2. a) $r = 0.05$, $T = 1$. Let $\tilde S^0_t = (1+r)^t$ for $t \in \{0,1\}$ and $\tilde S^1_0 = s_0 = 80$ and $\tilde S^1_1 = \begin{cases} 120 & \text{with prob. } 0.2 \\ 90 & 0.3 \\ 60 & 0.5 \end{cases}$

Hence the discounted process: $S^0_t \equiv 1$, $S^1_0 = s_0 = 80$ and $S^1_1 = (1+r)^{-1} \tilde S^1_1$

By assumption of (NA) $\Rightarrow M_e(\tilde S^0) \neq \emptyset$ (FTAP) so let $Q \in M_e(\tilde S^0)$. For Q to be an EMM we must have $E_Q[S^1_1 \mid \mathcal{F}_0] = S^1_0 = 80$, $Q[S^1_1 \cdot (1+r) \in \{120, 90, 60\}] = 1$ and $Q[S^1_1(1+r) = \begin{smallmatrix}120\\90\\60\end{smallmatrix}] > 0$.

- $E_Q[S^1_1 \mid \mathcal{F}_0] = E_Q[S^1_1] = E_Q[\tilde S^1_1 (1+r)^{-1}] = (1+r)^{-1}[q_u 120 + q_m 90 + q_d 60] \overset{!}{=} 80$

  $\Leftrightarrow q_u 120 + q_m 90 + q_d 60 = 80(1+r) = 84$. Since $q_u + q_m + q_d \overset{!}{=} 1$ and $q_u, q_m, q_d \overset{!}{>} 0$

  $\Rightarrow q_u 120 + (1 - q_u - q_d) 90 + q_d 60 = q_u 30 - q_d 30 + 90 = 84 \Leftrightarrow 30 q_u - 30 q_d = -6$

  $\Leftrightarrow q_u - q_d = -\frac{1}{5}$ so $q_d = q_u + \frac{1}{5}$ and $q_m = 1 - q_d - q_u = \frac{4}{5} - 2q_u$

  Additionally:
  - $0 < q_m < 1 \Leftrightarrow 0 < \frac{4}{5} - 2q_u < 1 \Leftrightarrow q_u < \frac{2}{5}$ and $-\frac{1}{10} < q_u$
  - $0 < q_d < 1 \Leftrightarrow 0 < q_u + \frac{1}{5} < 1 \Leftrightarrow -\frac{1}{5} < q_u$ and $q_u < \frac{4}{5}$
  - $0 < q_u < 1$

  Hence $M_e(\tilde S^0) = \{(q_u, \frac{4}{5} - 2q_u, q_u + \frac{1}{5}) \mid 0 < q_u < \frac{2}{5}\}$

By Thrm. 5.29 we know that $\Pi(H) = \{ E_Q[H] \mid Q \in M_e(\tilde{S}^0) \}$ so take $Q \in M_e(\tilde{S}^0)$:

$$E_Q[H] = (1+r)^{-1} E[(\tilde{S}_1^1 - 80)^+] = (1+r)^{-1} [q_u 40 + (\tfrac{4}{5} - 2q_u) 10 + (q_u + \tfrac{1}{5}) \cdot 0]$$

$$= (1+r)^{-1} [40 q_u + 8 - 20 q_u] = (1+r)^{-1} [20 q_u + 8]$$

So $\Pi(H) = \{ (1+r)^{-1} [20 q_u + 8] \mid 0 < q_u < \tfrac{2}{5} \} = [(1+r)^{-1} 8, (1+r)^{-1} 16] = [7.62, 15.24]$

b) Let H be a discounted & attainable claim $\Rightarrow \exists \xi$ self-financing and predictable such that $H = V_0 + (\xi \cdot S)_1 = V_0 + \xi_1 (S_1^1 - S_0^1)$ for some $V_0 \in \mathbb{R}$. ξ_1 is $\mathcal{F}_0 = \{\emptyset, \Omega\}$-meas. $\Rightarrow \xi_1 \in \mathbb{R}$

So $\{$claims $C \mid C$ attainable & discounted$\} = \{ V_0 + \xi (S_1^1 - S_0^1) \mid V_0, \xi \in \mathbb{R} \}$

c) Assume $\tilde{H}$ is attainable then Thrm. 5.32 implies $|\Pi(H)| = 1$ which contradicts a).

3. a) Let $M_0 = A_0 = 0$ P-a.s. and define $A_k := A_{k-1} + E[X_k - X_{k-1} \mid \mathcal{F}_{k-1}]$ for $k \geq 1$. $(A_k)_{k \in \mathbb{N}}$ is uniquely defined as we know A_0. A is integrable by construction.

Claim: A is adapted.

Proof: $A_1 = E[X_1 - X_0 \mid \mathcal{F}_0]$ which is by def. of cond. exp. $\mathcal{F}_0$-measurable. Assume A_k is $\mathcal{F}_{k-1}$-meas. $\Rightarrow A_{k+1} = A_k + E[X_{k+1} - X_k \mid \mathcal{F}_k]$ is $\mathcal{F}_k$-meas. since $E[X_{k+1} - X_k \mid \mathcal{F}] \in \mathcal{F}_k$ and $A_k \in \mathcal{F}_{k-1} \subset \mathcal{F}_k$ □

Define now $M_k := X_k - X_0 - A_k$ which is integrable since A and X are and indeed $M_0 = 0$ P-a.s.

We check: $E[M_{t+1} - M_t \mid \mathcal{F}_t] = E[X_{t+1} - A_{t+1} - X_t + A_t \mid \mathcal{F}_t] = E[X_{t+1} - X_t \mid \mathcal{F}_t] - A_{t+1} + A_t$

$$= A_{t+1} - A_t - A_{t+1} + A_t = 0$$

So M is a martingale and by construction $X_k = X_0 + M_k + A_k$ P-a.s. for $k \geq 0$.

b) As in a): $A_0 = 0$ P-a.s. $\Rightarrow A_k = A_{k-1} + E[X_k - X_{k-1} \mid \mathcal{F}_{k-1}]$ is P-a.s. unique by construction.

Hence $M_k = X_k - X_0 - A_k$ is also P-a.s. unique.

c) X is a super-martingale $\Leftrightarrow E[X_{t+1} \mid \mathcal{F}_t] \leq X_t \Leftrightarrow E[X_0 + M_{t+1} + A_{t+1} \mid \mathcal{F}_t] \leq X_0 + M_t + A_t$

$\Leftrightarrow X_0 + E[M_{t+1} \mid \mathcal{F}_t] + A_{t+1} \leq X_0 + M_t + A_t \Leftrightarrow M_t + A_{t+1} \leq M_t + A_t \Leftrightarrow A_{t+1} \leq A_t$