ETH Zürich
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Introduction to Mathematical Finance

Exercise sheet 8

Exercise 8.1 Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathcal{P})$ be a filtered probability space. Prove the *Optional Sampling Theorem* in the discrete time case: Let $(X_n)_{n\geq 0}$ be a martingale and τ a stopping time. Then

(a) The stopped process $(X_{n\wedge\tau})_{n\geq 0}$ is a martingale.

(b) If $P[\tau < \infty] = 1$ and $(X_{n\wedge\tau})_{n\geq 0}$ is uniformly integrable, then $E[X_\tau] = E[X_0]$.

Exercise 8.2 Let M be a *local martingale* which is bounded from below by $-a$ for some $a \geq 0$ and is integrable at the initial time: $M_0 \in L^1(P)$. Show from the definitions that M is a supermartingale.

Exercise 8.3 An *American option* with maturity T and payoff process $U = (U_k)_{k=0,\dots,T}$, where U is an adapted process, is a contract between buyer and seller where the buyer has the right to stop the contract at any time $0 \leq k \leq T$ and then to receive the (discounted) payoff U_k. The buyer is allowed to choose as exercise time for the option any stopping time with values in $\{0, \dots, T\}$. The goal of this exercise is to analyze the corresponding *arbitrage-free price* of an American option. With some effort, one can show that the *arbitrage-free price process* $\overline{V} = (\overline{V}_k)_{k=0,\dots,T}$ for an American option can be expressed by the backward recursive scheme

$$\overline{V}_T = U_T,$$

$$\overline{V}_k = \max\left\{U_k, E_Q\left[\overline{V}_{k+1}\,\middle|\,\mathcal{F}_k\right]\right\} \quad \text{for } k = 0, \dots, T-1, \tag{1}$$

where $\mathbb{Q}$ is an equivalent martingale measure for the considered market.

(a) Give an economic argument why (1) is a reasonable.

(b) Show that $\overline{V}$ is the smallest Q-supermartingale dominating U, i.e., show that

1. $\overline{V}$ is a Q-supermartingale such that $\overline{V}_k \geq U_k$ P-a.s. for all $k = 0, \dots, T$.
2. if V' is a Q-supermartingale such that $V'_k \geq U_k$ P-a.s. for all $k = 0, \dots, T$, then $V'_k \geq \overline{V}_k$ P-a.s. for all $k = 0, \dots, T$.

(c) Assume now that $r > 0$ so that the bank account is strictly increasing.

1. Show that in the *put option* case, i.e., $U_j = \frac{1}{(1+r)^j}(\widetilde{K} - \widetilde{S}^1_j)^+$, the price of an American option at time 0 is greater that the price of a European option, for large enough strikes $\widetilde{K}$, i.e.,

$$\overline{V}_0 > V_0^{\widetilde{P}^{\widetilde{K}}_T},$$

for $\widetilde{K}$ large enough, where $V_0^{\widetilde{P}^{\widetilde{K}}_T}$ denotes the discounted price at time 0 of a European put option with maturity T and strike price $\widetilde{K}$.

2. Show that in the *call option* case, i.e., $U_j = \frac{1}{(1+r)^j}(\widetilde{S}^1_j - \widetilde{K})^+$, the price of the American call option and the European call option coincide. This means, show that

$$\overline{V}_0 = V_0^{\widetilde{C}^{\widetilde{K}}_T},$$

where $V_0^{\widetilde{C}^{\widetilde{K}}_T}$ denotes the price at time 0 of an European call option with maturity T and strike price $\widetilde{K}$.

Exercise 8.4 Python - American option

(a) Change the trinomial price function to also handle American options.

(b) Change the trinomial price function to also handle barrier conditions.