Introduction to Mathematical Finance

Solution sheet 8

Solution 8.1 Denote by $(\tau_n)_{n\in\mathbb{N}}$ a localizing sequence and let $Y = M + a$ so that $Y \geq 0$ P-a.s. Then $Y^{\tau_n} 1_{\{\tau_n > 0\}}$ is a nonnegative martingale for every $n \in \mathbb{N}$. Note that since Y_0 is integrable, we can even drop the indicator function. Indeed,

$$Y_k^{\tau_n} = Y_0 1_{\{\tau_n = 0\}} + Y_k^{\tau_n} 1_{\{\tau_n > 0\}},$$

where the first term is integrable, $\mathcal{F}_0$-measurable, and constant in k, hence a martingale.

Moreover, since $\tau_n \nearrow \infty$ P-a.s.

$$\lim_{n\to\infty} Y_k^{\tau_n} = Y_k \quad P\text{-a.s.},$$

for every $k \in \mathbb{N}_0$. Condition (M1) is satisfied by assumption. By Fatou's lemma,

$$E[Y_k] = E\left[\liminf_{n\to\infty} Y_k^{\tau_n}\right] \leq \liminf_{n\to\infty} E[Y_k^{\tau_n}] = \liminf_{n\to\infty} E[Y_0^{\tau_n}] = E[Y_0] < \infty,$$

establishing (M2). Similarly,

$$\begin{aligned} E[Y_k|\mathcal{F}_j] &= E\left[\liminf_{n\to\infty} Y_k^{\tau_n}|\mathcal{F}_j\right] \\ &\leq \liminf_{n\to\infty} E[Y_k^{\tau_n}|\mathcal{F}_j] = \liminf_{n\to\infty} Y_j^{\tau_n} = Y_j \quad P\text{-a.s.} \end{aligned}$$

for all $j \leq k$, showing that Y, and therefore also M, is indeed a supermartingale.

Solution 8.2

(a) Suppose we are at time $T-1$. We have two possibilities; either we exercise the option immediately, in which case we get the *payoff* U_{T-1}, or we don't exercise the option, in which case the *price* at time $T-1$ of the payoff U_T is given by its Q-conditional expectation given $\mathcal{F}_{T-1}$. Naturally, one takes the maximum of these two possibilities. The price at time k is argued similarly when we admit that $\overline{V}_{k+1}$ is the reasonable price at time $k+1$.

(b) 1. In order to show that $\overline{V}$ is a Q-supermartingale dominating U, we have to check whether
   (a) $\overline{V}$ is Q-integrable, adapted and dominates U.
   (b) it satisfies the Q-supermartingale property for all $k = 0, \dots, T$, i.e.,

$$\overline{V}_k \geq E_Q\left[\overline{V}_{k+1} \,\middle|\, \mathcal{F}_k\right] \quad P\text{-a.s.}$$

   We argue (i) and (ii) inductively. By assumption, U is adapted, hence $\overline{V}_T$ is $\mathcal{F}_T$-measurable. Since $\overline{V}_k$ is the maximum of two $\mathcal{F}_k$-measurable random variables, it is itself $\mathcal{F}_k$-measurable. Hence, $\overline{V}$ is adapted. The integrability is trivially satisfied since we work with a finite probability space.
   Next, $\overline{V}_T = U_T$, and we obtain directly from the definition of $\overline{V}_k$ that

$$\overline{V}_k = \max\left\{U_k, E_Q\left[\overline{V}_{k+1} \,\middle|\, \mathcal{F}_k\right]\right\} \geq U_k\,.$$

   Hence, $\overline{V}$ dominates U.
   Finally, for the Q-supermartingale property, we simply use the definition of $\overline{V}$, which yields

$$\overline{V}_k \geq E_Q\left[\overline{V}_{k+1} \,\middle|\, \mathcal{F}_k\right]\,.$$

2. In order to prove the minimality property of $\overline{V}$, let V' be a Q-supermartingale dominating U. This includes the two inequalities

$$V'_k \geq U_k \quad P\text{-a.s.} \quad \text{and} \quad V'_k \geq E_Q\left[\, V'_{k+1} \,\middle|\, \mathcal{F}_k\right] ,$$

which together yield $V'_k \geq \max\left\{U_k, E_Q\left[\, V'_{k+1} \,\middle|\, \mathcal{F}_k\right]\right\}$. Since $V'_T \geq U_T = \overline{V}_T$, we conclude that

$$V'_{T-1} \geq \max\left\{U_{T-1}, E_Q\left[V'_T \,\middle|\, \mathcal{F}_{T-1}\right]\right\} \geq \max\left\{U_{T-1}, E_Q\left[\overline{V}_T \,\middle|\, \mathcal{F}_{T-1}\right]\right\} = \overline{V}_{T-1} .$$

This implies that $V'_{T-1} \geq \overline{V}_{T-1}$. Assuming that $V'_{k+1} \geq \overline{V}_{k+1}$ P-a.s., we can repeat the same argument replacing $T-1$ by $k+1$ which finally yields the desired inequality $V'_{k+1} \geq \overline{V}_{k+1}$ P-a.s..

(c) 1. By definition of $\overline{V}$, we see that the value of $\overline{V}_{T-1}$ is greater or equal to the value of a European put option at time $T-1$, i.e.,

$$\overline{V}_{T-1} \geq E_Q\left[\overline{V}_T \,\middle|\, \mathcal{F}_{T-1}\right] .$$

If we inductively assume that

$$\overline{V}_{k+1} \geq V_{k+1}^{\widetilde{P_T^{\widetilde{K}}}},$$

we conclude that

$$\begin{aligned}
\overline{V}_k &= \max\left\{U_k, E_Q\left[\overline{V}_{k+1} \,\middle|\, \mathcal{F}_k\right]\right\} \\
&\geq \max\left\{U_k, E_Q\left[V_{k+1}^{\widetilde{P_T^{\widetilde{K}}}} \,\middle|\, \mathcal{F}_k\right]\right\} \\
&= \max\left\{U_k, V_k^{\widetilde{P_T^{\widetilde{K}}}}\right\} \\
&\geq V_k^{\widetilde{P_T^{\widetilde{K}}}} .
\end{aligned}$$

Thus, we also have $\overline{V}_0 \geq V_0^{\widetilde{P_T^{\widetilde{K}}}}$. We show that for certain strike prices $\widetilde{K}$, we cannot have equality. To that end, we focus on one period. There, the price of a European put option, respectively of an American put, at time 0 is given by

$$E_0 := E_Q\left[\left(\frac{K}{1+r} - S_1^1\right)^+\right], \quad \text{respectively} \quad A_0 := \overline{V}_0 = \max\left\{E_0, (K - S_0^1)^+\right\} .$$

For simplicity, we assume $S_0^1 = 1$. Then E_0 can be computed as

$$E_0 = q\left(\frac{K}{1+r} - \frac{1+u}{1+r}\right)^+ + (1-q)\left(\frac{K}{1+r} - \frac{1+d}{1+r}\right)^+ .$$

Let $K > \max\{u, 1\}$, so that $\frac{K}{1+r} - \frac{1+u}{1+r} > 0$. Then, we get

$$E_0 = \frac{K}{1+r} - q\frac{1+u}{1+r} - (1-q)\frac{1+d}{1+r} = \frac{K}{1+r} - 1 .$$

because $E_Q\left[S_1^1\right] = S_0^1 = 1$. On the other hand, for the American option price, we have that

$$A_0 = \max\left\{E_0, (K - S_0^1)^+\right\} = K - 1 ,$$

Hence, we see that $A_0 > E_0$.

2. We claim that the process

$$\left(\left(S_j^1 - \frac{\widetilde{K}}{(1+r)^j}\right)^+\right)_{j=0,\dots,T} \quad \text{is a } Q\text{-submartingale.}$$

To that end, we only have to verify the Q-submartingale property. So let us consider some $k = 0, \dots T-1$ and use the known properties

$$E_Q\left[\left(S_{k+1}^1 - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+ \middle| \mathcal{F}_k\right] \overbrace{\geq}^{\text{Jensen}} \left(E_Q\left[S_{k+1}^1 \,\middle|\, \mathcal{F}_k\right] - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+$$

$$\underbrace{\geq}_{\text{martingale property}} \left(S_k^1 - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+$$

$$\underbrace{\geq}_{(*)} \left(S_k^1 - \frac{\widetilde{K}}{(1+r)^{k}}\right)^+ ,$$

where in $(*)$ we have used the fact that

$$K \mapsto (x-K)^+$$

is decreasing, hence

$$\left(S_k^1 - \frac{\widetilde{K}}{(1+r)^{k}}\right)^+ \leq \left(S_k^1 - \frac{\widetilde{K}}{(1+r)^{k+1}}\right)^+ ,$$

because $r \geq 0$. By the Q-submartingale property, we have for all $j = 0, \dots, T$ that

$$U_j = \left(S_j^1 - \frac{\widetilde{K}}{(1+r)^j}\right)^+ \leq E_Q\left[\left(S_{j+1}^1 - \frac{\widetilde{K}}{(1+r)^{j+1}}\right)^+ \middle| \mathcal{F}_j\right] = E_Q\left[U_{j+1} \,\middle|\, \mathcal{F}_j\right] .$$

For $j = T-1$ this gives

$$\overline{V}_{T-1} = \max\{U_{T-1}, E_Q\left[U_T \,\middle|\, \mathcal{F}_{T-1}\right]\} = E_Q\left[\overline{V}_T \,\middle|\, \mathcal{F}_{T-1}\right] .$$

The same induction argument finally yields that $\overline{V}_k = E_Q\left[\overline{V}_{k+1} \,\middle|\, \mathcal{F}_k\right]$, i.e., $\overline{V}$ is a Q-martingale with terminal value $\left(S_T^1 - \frac{\widetilde{K}}{(1+r)^T}\right)^+$. Thus, we obtain

$$\overline{V}_0 = V_0^{C_T^K} .$$

Solution 8.3

```
def trinomial_price(maturity, spot, strike, rate, vol, steps_number, payoff_fct
    =None,barrier_condition=None, is_american=False, graph_name=None):

  if not barrier_condition:
    barrier_condition = lambda unused_spot: True

```

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{...}

 # The following two list are only needed to display the graph:
 spot_prices_history = [spot_prices]
 option_prices_history = [option_prices]

 def next_option_price(spot_price, price_up, price_midlle, price_down):
   if barrier_condition(spot_price):
     option_price = (discount_factor *(proba_up * price_up +
                               proba_middle * price_midlle +
                               proba_down * price_down))
     if is_american:
         exercise = payoff_fct(spot_price, strike)
         if option_price < exercise: option_price = exercise
   else:
     option_price = 0
   return option_price

 while len(option_prices) > 1:
   option_prices = [next_option_price(spot_prices[i], *option_prices[i-1:i+2])
                    for i in range(1, len(option_prices)-1)]
   spot_prices = spot_prices[1:-1]
   spot_prices_history.insert(0, spot_prices)
   option_prices_history.insert(0, option_prices)
 ..
 return option_prices[0]
```