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Exercise: (X_n), (Y_n) submartingales,
τ a stopping time s.t. $X_{\tau(\omega)}(\omega) \le Y_{\tau(\omega)}(\omega)$

$$Z_n(\omega) := \begin{cases} X_n(\omega) & \text{if } n < \tau(\omega) \\ Y_n(\omega) & \text{if } n \ge \tau(\omega) \end{cases}$$

Show: (Z_n) is a sub-martingale.

Observe that the process (Z_n) can be rewritten as

$$Z_n(\omega) = X_n(\omega) \mathbb{1}_{\{n < \tau(\omega)\}} + Y_n(\omega) \mathbb{1}_{\{n \ge \tau(\omega)\}}$$

- Since τ is a stopping time, the sets $\{\tau \le n\}$ and $\{\tau > n\} = \{\tau \le n\}^c$ are in $\mathcal{F}_n$.

  $\Rightarrow$ Hence, $\mathbb{1}_{\{n < \tau(\omega)\}}$ and $\mathbb{1}_{\{n \ge \tau(\omega)\}}$ are $\mathcal{F}_n$-measurable.

  $\Rightarrow$ Since X_n and Y_n are adapted as (X_n) and (Y_n) are sub-martingales, also Z_n is $\mathcal{F}_n$-measurable.
  So (Z_n) is adapted.

- Check that Z_n is integrable:

$$E[|Z_n|] = E[|X_n \mathbb{1}_{\{n<\tau\}} + Y_n \mathbb{1}_{\{n \ge \tau\}}|] \le E[|X_n \mathbb{1}_{\{n<\tau\}}|] + E[|Y_n \mathbb{1}_{\{n\ge\tau\}}|]$$

$$\le \underbrace{E[|X_n|]}_{\substack{<\infty \text{ as} \\ X_n \in L^1}} + \underbrace{E[|Y_n|]}_{\substack{<\infty \text{ as} \\ Y_n \in L^1}} < \infty$$

- It remains to check the conditional expectation property.

$$E[Z_{n+1} \mid \mathcal{F}_n] = E[X_{n+1} \underbrace{\mathbb{1}_{\{n+1<\tau\}}}_{= \mathbb{1}_{\{n<\tau\}} - \mathbb{1}_{\{n+1=\tau\}}} + Y_{n+1} \mathbb{1}_{\{n+1 \ge \tau\}} \mid \mathcal{F}_n]$$

$$= E[X_{n+1} (\underbrace{\mathbb{1}_{\{n<\tau\}}}_{\mathcal{F}_n\text{-mbl.}} - \mathbb{1}_{\{n+1=\tau\}}) \mid \mathcal{F}_n] + E[Y_{n+1} \mathbb{1}_{\{n+1 \ge \tau\}} \mid \mathcal{F}_n]$$

$$= E[X_{n+1} \mid \mathcal{F}_n] \mathbb{1}_{\{n<\tau\}} - E[X_{n+1} \mathbb{1}_{\{n+1=\tau\}} \mid \mathcal{F}_n] + E[Y_{n+1} \mathbb{1}_{\{n+1 \ge \tau\}} \mid \mathcal{F}_n]$$

$$\overset{X \text{ submart.}}{\ge} X_n \mathbb{1}_{\{n<\tau\}} + E[\underbrace{(-X_\tau)}_{\ge -Y_\tau} \mathbb{1}_{\{n+1=\tau\}} \mid \mathcal{F}_n] + E[Y_{n+1} \mathbb{1}_{\{n+1 \ge \tau\}} \mid \mathcal{F}_n]$$

$$\ge X_n \mathbb{1}_{\{n<\tau\}} + E[Y_{n+1} (\mathbb{1}_{\{n+1 \ge \tau\}} - \mathbb{1}_{\{n+1=\tau\}}) \mid \mathcal{F}_n]$$

$$= X_n \mathbb{1}_{\{n<\tau\}} + E[\underbrace{Y_{n+1}}_{\text{mart.}} \underbrace{\mathbb{1}_{\{n \ge \tau\}}}_{\mathcal{F}_n\text{-mbl.}} \mid \mathcal{F}_n] \ge X_n \mathbb{1}_{\{n<\tau\}} + Y_n \mathbb{1}_{\{n \ge \tau\}} = Z_n \text{ P-a.s.}$$