Coordinator
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Introduction to Mathematical Finance

Exercise sheet 9

Exercise 9.1 (X_n) a martingale, $n \geq 0$ and $E(X_n^2) < +\infty$.

(a) Let $\lambda > 0$,

$$\tau := \inf\{k \geq 0 : |X_k| \geq \lambda\} \qquad (\min \emptyset = \infty)$$

Show that $\tau \wedge n$ is a stopping time.

(b) $X_n^* := \max_{0 \leq k \leq n} |X_k|$, show that

$$\lambda P(X_n^* \geq \lambda) \leq E(|X_n| 1_{X_n^* > \lambda}) .$$

(c) Let $b > 0$, by writing $(X_n^* \wedge b)^2 = 2 \int_0^{X_n^* \wedge b} x dx$, show that

$$E[(X_n^* \wedge b)^2] \leq 2E[(X_n^* \wedge b)|X_n|] .$$

(d) Using the Cauchy-Schwartz inequality, show that

$$E[X_n^*] < +\infty$$

and that

$$E[\sup_{0 \leq k \leq n} X_k^2] \leq 4E[X_n^2] .$$

Exercise 9.2 Let

$$H_t^K := \frac{(K - S_t)_+}{(1+r)^t}$$

be the discounted payoff of an American put option with strike K in a market model with one risky asset $S = (S_t)_{t=0,\ldots T}$ and a riskless asset $S_t^0 = (1+r)^t$, where $r > 0$. We denote by $\tau_{\min}^K$ the minimal stopping time of the buyer's problem to maximize $E[H_t^K]$ over $\tau \in \mathcal{T}$.

(a) Show that $\tau_{\min}^K \geq \tau_{\min}^{K'}$ P-a.s. if $K \leq K'$.

(b) Show that ess $\inf_{K \geq 0} \tau_{\min}^K = 0$ P-a.s.

(c) Use (b) and the fact that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ to conclude that there exists $K_0 \geq 0$ such that $\tau_{\min}^K = 0$ P-a.s. for all $K \geq K_0$.

Exercise 9.3 Show that in every arbitrage-free market model and for any discounted American claim H,

$$\inf_{P^* \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} E^*[H_\tau] < \infty$$

and that the set $\Pi(H)$ of arbitrage-free prices is nonempty.