IMF Serie 8 — Florian Krach

Exercise 9.1 (X_n) a martingale, $n \geq 0$ and $E(X_n^2) < +\infty$.

(a) Let $\lambda > 0$,
$$\tau := \inf\{k \geq 0 : |X_k| \geq \lambda\} \qquad (\min \emptyset = \infty)$$
Show that $\tau \wedge n$ is a stopping time.

(b) $X_n^* := \max_{0 \leq k \leq n} |X_k|$, show that
$$\lambda P(X_n^* \geq \lambda) \leq E(|X_n| 1_{X_n^* \geq \lambda}).$$

(c) Let $b > 0$, by writing $(X_n^* \wedge b)^2 = 2\int_0^{X_n^* \wedge b} x\,dx$, show that
$$E[(X_n^* \wedge b)^2] \leq 2E[(X_n^* \wedge b)|X_n|].$$

(d) Using the Cauchy-Schwartz inequality, show that
$$E[X_n^*] < +\infty$$
and that
$$E[\sup_{0 \leq k \leq n} X_k^2] \leq 4E[X_n^2].$$

a.) Proof: $\leadsto$ let $k \in \mathbb{N}$ $\leadsto$ if $k \geq n \Rightarrow \{\tau \wedge n \leq k\} = \Omega \in \mathcal{F}_0 \subset \mathcal{F}_k$

$\leadsto$ if $k < n \Rightarrow \{\tau \wedge n \leq k\} = \{\tau \leq k\} = \bigcup_{j=0}^{k} \underbrace{[|X_j| \geq \lambda]}_{\in \mathcal{F}_j \subset \mathcal{F}_k \text{, since } X \text{ adapted}} \in \mathcal{F}_k$

$\Rightarrow \{\tau \wedge n \leq k\} \in \mathcal{F}_k \quad \forall k \in \mathbb{N}$

$\Rightarrow \tau \wedge n$ is a stopping time ☑

b.) Proof: $\leadsto (X_n)_n$ mg. $\Rightarrow (|X_n|)_n$ is a submartingale (since $x \mapsto |x|$ is convex & using Jensen UGL)

$\leadsto$ therefore we can use Doob's inequality (see Prob. Theory (3.6.1)) for $(|X_k|)_k$ which directly gives:

$\lambda \cdot P(\max_{0 \leq k \leq n} |X_k| \geq \lambda) \leq E(|X_n| \cdot 1_{\{\max_{0 \leq k \leq n} |X_k| \geq \lambda\}})$ ☑

c.) Proof: $E((X_n^* \wedge b)^2) = E\left(2\int_0^{X_n^* \wedge b} x\,dx\right) = E\left(2 \cdot \int_0^\infty x \cdot 1_{\{X_n^* \wedge b \geq x\}}\,dx\right) \overset{\text{Fubini}}{=} 2 \cdot \int_0^\infty E(x \cdot 1_{\{X_n^* \wedge b \geq x\}})\,dx =$

$= 2 \cdot \int_0^\infty x \cdot P(X_n^* \wedge b \geq x)\,dx \leq 2 \cdot \int_0^\infty x \cdot \left(P(\underbrace{[X_n^* \wedge b \geq x] \cap \{x \leq b\}}_{= [X_n^* \geq x] \cap \{x \leq b\} \text{ independent}}) + P(\underbrace{[X_n^* \wedge b \geq x] \cap \{x > b\}}_{= \emptyset})\right)dx = 2 \cdot \int_0^\infty \underbrace{x \cdot P([X_n^* \geq x])}_{\leq E(|X_n| \cdot 1_{X_n^* \geq x})} \cdot \underbrace{P(x \leq b)}_{= E(1_{\{x \leq b\}})}\,dx \overset{\text{as before}}{=}$

$= 2 \cdot \int_0^\infty E(|X_n| \cdot 1_{X_n^* \wedge b \geq x}) \overset{\text{Fubini}}{=} 2 \cdot E\left(\int_0^\infty |X_n| \cdot 1_{\{X_n^* \wedge b \geq x\}}\,dx\right) = 2 \cdot E\left(|X_n| \cdot \int_0^{X_n^* \wedge b} dx\right) = 2 \cdot E(|X_n| \cdot (X_n^* \wedge b))$ ☑

d.) Proof: $\leadsto X_n^* \leq |X_0| + \ldots + |X_n| \Rightarrow E(X_n^*) \leq \sum_{i=0}^{n} \overbrace{E(|X_i|)}^{< \infty} < \infty$ ✓

$\leadsto$ from c.) we have: $E(|(X_n^* \wedge b)|^2) \overset{c)}{\leq} 2 \cdot E(|X|_n \cdot (X_n^* \wedge b)) \overset{\text{Hölder}}{\leq} 2 \cdot E(|X_n|^2)^{1/2} \cdot E(|(X_n^* \wedge b)|^2)^{1/2} \overset{(*)}{\Leftrightarrow}$

(*) if LHS $= 0$ the following equivalence is clear since $|X_n|^2 \geq 0$; if LHS > 0 then we can divide

$\Leftrightarrow E((X_n^* \wedge b)^2)^{1/2} \leq 2 \cdot E(|X_n|^2)^{1/2} \Leftrightarrow E((X_n^* \wedge b)^2) \leq 4 \cdot E(|X_n|^2) \quad (1)$

$\leadsto$ by monotone convergence we have: $\lim_{b \to \infty} E((X_n^* \wedge b)^2) = E(\lim_{b \to \infty} (X_n^* \wedge b)^2) = E(X_n^{*2}) \quad (2)$

$\leadsto (1) \& (2) \Rightarrow E(\sup_{0 \leq k \leq n} X_k^2) = E(X_n^{*2}) = \lim_{b \to \infty} E((X_n^* \wedge b)^2) \leq 4 \cdot E(|X_n|^2)$ ☑

Exercise 9.2 Let
$$H_t^K := \frac{(K - S_t)_+}{(1+r)^t}$$
be the discounted payoff of an American put option with strike K in a market model with one risky asset $S = (S_t)_{t=0,\ldots T}$ and a riskless asset $S_t^0 = (1+r)^t$, where $r > 0$. We denote by $\tau_{\min}^K$ the minimal stopping time of the buyer's problem to maximize $E[H_\tau^K]$ over $\tau \in \mathcal{T}$.

(a) Show that $\tau_{\min}^K \geq \tau_{\min}^{K'}$ P-a.s. if $K \leq K'$. — in FS Ex 6.24

(b) Show that $\operatorname{ess\,inf}_{K \geq 0} \tau_{\min}^K = 0$ P-a.s.

(c) Use (b) and the fact that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ to conclude that there exists $K_0 \geq 0$ such that $\tau_{\min}^K = 0$ P-a.s. for all $K \geq K_0$

a.) Proof: $\leadsto$ for $K \leq K'$ we have: $(K - S_t)^+ \leq (K' - S_t)^+$

$\Rightarrow H_t^K \leq H_t^{K'} \quad \forall t$

$\leadsto$ let $U^K, U^{K'}$ be the Snell-envelopes for $H^K, H^{K'}$ w.r.t. P

$\Rightarrow U_T^K = H_T^K$, $U_t^K = H_t^K \vee E(U_{t+1}^K | \mathcal{F}_t)$ (see FS after Rem 6.17)

$\leadsto$ by def. (FS after Rem 6.17): $\tau_{\min}^K = \inf\{t \geq 0 | U_t^K = H_t^K\}$

$\Rightarrow U_{\tau_{\min}^K}^K = H_{\tau_{\min}^K}^K$ and therefore: $E(U_{\tau_{\min}^K + 1}^K | \mathcal{F}_\tau) \leq H_{\tau_{\min}^K}^K$

$\leadsto U^{K'}$ is a superm.g with $U_t^{K'} \geq H_t^{K'} \geq H_t^K$ $\leadsto$ since U^K is the smallest superm.g satisfying $U_t^K \geq H_t^K$ we have: $U_t^{K'} \geq U_t^K$

Assume: $\Omega^* := \{\tau_{\min}^K < \tau_{\min}^{K'}\}$ has $P(\Omega^*) > 0$ $\leadsto$ let $\Omega_t^* := \Omega^* \cap \{\tau_{\min}^K = t\}$ $\Rightarrow \exists t: P(\Omega_t^*) > 0$ $\leadsto t < T$

$\leadsto$ on this set Ω_t^* we have: $U_t^K = H_t^K$ and $U_{t'}^{K'} > H_{t'}^{K'} \quad \forall t' \leq t \Rightarrow U_{t'}^{K'} = E(U_{t'+1}^{K'} | \mathcal{F}_t) > H_{t'}^{K'} \quad \forall t' \leq t$

b.) Proof: $\leadsto \operatorname{ess\,inf}_{K \geq 0} \tau_{\min}^K \overset{P\text{-a.s.}}{\leq} \tau_{\min}^0 = \inf\{t \geq 0 | U_t^0 = H_t^0\}$ (1) $\leadsto H_t^0 = \frac{(-S_t)^+}{(1+r)^t} \overset{S_t \geq 0}{=} 0 \quad \forall t$ $\leadsto U_t^0 = \operatorname{ess\,sup}_{t \leq \tau \leq T} E(\overset{=0}{H_\tau^0} | \mathcal{F}_t) = 0$

$\Rightarrow \forall t: H_t^0 = U_t^0 \Rightarrow \tau_{\min}^0 = 0 \overset{(1)}{\Rightarrow} \operatorname{ess\,inf}_{K \geq 0} \tau_{\min}^K = 0$ P-a.s. ☑

c.) Proof: $\leadsto$ from b) we already know that $K_0 = 0$ fulfils the claim, since $\tau_{\min}^0 = 0$ and using a.) ☑

Exercise 9.3 Show that in every arbitrage-free market model and for any discounted American claim H, assume that (*) holds

$$\inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} E^*[H_\tau] \; < \infty$$

and that the set $\Pi(H)$ of arbitrage-free prices is nonempty.

Proof: Claim1: $\inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}_{P^*}(H_\tau) < \infty$

Proof1: $\rightsquigarrow$ if $\inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) < \sup_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) \leq \infty$

then the claim obviously holds $\Rightarrow$ we can assume that $\inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) = \sup_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau)$ (1)

$\rightsquigarrow$ (as in Thm 6.31), we have to assume that: $\exists\, Q\in\mathcal{P}: \forall t\in\{0,..,T\}: H_t\in L^1(Q)$ (*)

$\Rightarrow \mathbb{E}_Q(|H_t|) < \infty \quad \forall t$ (3)

$\rightsquigarrow$ we obviously have $|H_\tau| \leq |H_0| + \ldots + |H_T|$ $\forall$ stopping times $0\leq\tau\leq T$ (2)

$\rightsquigarrow$ using (1) we have: $\inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) \overset{(1)}{=} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q(H_\tau) \overset{(2)}{\leq} \mathbb{E}\left(\sum_{i=0}^{T} |H_i|\right) \overset{(3)}{<} \infty$ $\square$ (Claim1)

Claim2: $\Pi(H)\neq\emptyset$, Proof2: Case1: $\pi := \inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) \overset{!}{=} \sup_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau)$ $\rightsquigarrow$ we check that π satisfies Def 6.29

$\rightsquigarrow$ let τ^* be an optimal stopping time for H wrt. $P^*\in\mathcal{P}$, i.e. $\mathbb{E}^*(H_{\tau^*}) = \sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau)$ $\rightsquigarrow$ we can e.g. choose $\tau^* = \tau_{min}$, which exists (since $\tau_{min}\leq T$) and is optimal by Thm 6.18

$\Rightarrow \pi = \sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) = \mathbb{E}^*(H_{\tau^*}) =: \pi^*$ $\Rightarrow$ (i) of Def 6.29 fulfilled

$\rightsquigarrow$ (ii) of Def 6.29: $\nexists\, \tau'\in\mathcal{T}\ \forall \pi'\in\Pi_{E0}(H_{\tau'}): \pi<\pi'$ $\iff$ $\forall\tau'\in\mathcal{T}\ \exists\,\pi'\in\Pi_{E0}(H_{\tau'}): \pi\geq\pi'$ (4)

$\rightsquigarrow$ let $\tau'\in\mathcal{T} \Rightarrow \pi = \sup_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau) \geq \mathbb{E}_{P^*}(H_{\tau'})\ \forall P^*\in\mathcal{P} \overset{(4)}{\Rightarrow}$ (ii) of Def 6.29 fulfilled $\Rightarrow \pi\in\Pi(H)$ $\square$ (Case1)

Case2: $\underbrace{\inf_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau)}_{=a} < \pi < \underbrace{\sup_{P^*\in\mathcal{P}}\sup_{\tau\in\mathcal{T}} \mathbb{E}^*(H_\tau)}_{=b}$, where π is an arbitrary element between LHS & RHS $\rightsquigarrow$ we check that π satisfies Def 6.29

$\rightsquigarrow$ by the def of supremas we find a $P^*\in\mathcal{P}$ and $\tau\in\mathcal{T}$ s.t. $\pi < \mathbb{E}_{P^*}(H_\tau)$ $\Rightarrow$ (i) of Def 6.29 fulfilled

$\rightsquigarrow$ by the def of sup and inf we find a P^* s.t. $\pi > \sup_{\tau\in\mathcal{T}} \mathbb{E}_{P^*}(H_\tau) \overset{(4)}{\Rightarrow}$ (ii) of Def 6.29 fulfilled

$\Rightarrow \pi\in\Pi(H)$ and i.e. $(a,b)\subset\Pi(H)$ $\square$ (Case2) $\square$ (Claim2) $\blacksquare$