Introduction to Mathematical Finance

Solution sheet 9

Solution 9.1

(a) $\{\tau \leq k\} = \bigcup_{i=0}^{k} \{|X_i| > \lambda\}$. The finite countable union of events $\in \mathcal{F}_k$ is $\mathcal{F}_k$-measurable. so $\{\tau \leq k\} \in \mathcal{F}_k$, thus τ is a stopping time. $\{\tau \wedge n \leq n\} = \{\tau \leq n\} \in \mathcal{F}_n$, so $\tau \wedge n$ is a stopping time.

(b)

$$\begin{aligned}
E(|X_n| 1_{\{X_n^* > \lambda\}}) &= E(|X_n| 1_{\{\tau \leq n\}}) \\
&= E(E(|X_n| 1_{\{\tau \leq n\}} \ |\mathcal{F}_{\tau \wedge n})) \quad \text{tower property} \\
&= E(1_{\{\tau \leq n\}} E(|X_n| \ |\mathcal{F}_{\tau \wedge n})) \quad \text{because } \ \{\tau \leq n\} \in \mathcal{F}_n \\
&\geq E(1_{\{\tau \leq n\}} |E(X_n \ |\mathcal{F}_{\tau \wedge n})|) \quad \text{Jensen with } \ f(x) = |x| \\
&= E(1_{\{\tau \leq n\}} |X_{\tau \wedge n}|) \\
&= E(1_{\{\tau \leq n\}} |X_{\tau}|) \\
&\geq E(1_{\{\tau \leq n\}} \lambda) \quad \text{definition of the stopping time } \tau \\
&= \lambda P(\tau \leq n) \\
&= \lambda P(X_n^* > \lambda)
\end{aligned}$$

(c)

$$\begin{aligned}
E(X_n^* \wedge b)^2) &= E(2 \int_0^{X_n^* \wedge b} x dx) \\
&= 2E(\int_0^{+\infty} x 1_{\{x \leq X_n^* \wedge b\}} dx) \\
&= 2 \int_0^{+\infty} x E(1_{\{x \leq X_n^* \wedge b\}}) dx \quad \text{Fubini positif} \\
&= 2 \int_0^{+\infty} x P(x \leq X_n^* \wedge b) dx \\
&\leq 2 \int_0^{+\infty} E(|X_n \wedge b| 1_{\{x \leq X_n^* \wedge b\}}) dx \\
&\leq 2 \int_0^{+\infty} E(|X_n| 1_{\{x \leq X_n^* \wedge b\}}) dx \\
&\leq 2E(|X_n| \int_0^{+\infty} 1_{\{x \leq X_n^* \wedge b\}} dx) \\
&\leq 2E(|X_n| (X_n^* \wedge b))
\end{aligned}$$

(d) $E(X_n^*) = E(\sup_{0 \leq k \leq n} |X_k|) \leq \sup_{0 \leq k \leq n} E(|X_k|) < +\infty$, so $\sup_n E(X_n^*) < +\infty$.

We notice that $\sup_{0 \leq k \leq n} (X_k)^2 = (\sup_{0 \leq k \leq n} X_k)^2 = (X_n^*)^2$ and so that $E(\sup_{0 \leq k \leq n} (X_k)^2) = E((X_n^*)^2)$, so we have to show that

$$E((X_n^*)^2) \leq 4E(X_n^2).$$

For all $b > 0$,

$$E(X_n^* \wedge b)^2) \leq 2E(|X_n|(X_n^* \wedge b)) \leq 2E(|X_n|^2)^{1/2}E((X_n^* \wedge b)^2)^{1/2}$$

So by dividing by $E((X_n^* \wedge b)^2)^{1/2}$ because $X_n^* \wedge b \geq b > 0$ we obtain

$$E((X_n^* \wedge b)^2)^{1/2} \leq 2E(|X_n|^2)^{1/2}$$

So

$$E((X_n^* \wedge b)^2) \leq 4E(|X_n|^2)\,.$$

$\lim_{b\to\infty} X_n^* \wedge b = X_n^*$ and $|X_n^* \wedge b| \leq X_n^*$ and $\sup_n E(X_n^*) < +\infty$. By the dominated convergence theorem, $\lim_{b\to\infty} E(X_n^* \wedge b) = E(X_n^*)$ So

$$E((X_n^*)^2) \leq 4E(|X_n|^2).$$

Solution 9.2 *The statement of this exercise is true for the continuous time*[1]*.But it is no longer true for the discrete time. A counterexample can be found.* **This question is no more on the exam.**

(a) We introduce the non-negative random field

$$\varphi(t, K) = u_t(K) - K + S_t$$

[2]in term which we can rewrite

$$\tau_{\min}^K = \inf\{t \geq 0 \;\mid\; \varphi(t, K) = 0\ \}\,.$$

For a fixed $t \in [0, T]$, $K \to u_t(K)$ is convex and increasing. $K \to K - u_t(K)$ is also concave and increasing, so $K \to u_t(K) - K - S_t$ is decreasing. In other words $K \to \varphi(t, K)$ is decreasing. Thus for $K \leq K'$, we have

$$0 \leq \varphi(\tau_{\min}^K, K') \leq \varphi(\tau_{\min}^K, K) = 0\,.$$

So

$$\varphi(\tau_{\min}^K, K') = 0$$

hence

$$\tau_{\min}^{K'} \leq \tau_{\min}^K \quad P\,a.s.$$

(b) We have $\operatorname{ess\,inf}_{K\geq 0}(\tau_{\min}^K) \leq \tau_{\min}^0$ with $\tau_{\min}^0 = \inf\{t \geq 0 \;\mid\; U_t^0 = H_t^0\ \}$. As

$$H_t^0 := \frac{(0 - S_t)_+}{(1 + r)^t} = 0$$

and

$$U_t^0 = \operatorname{ess\,sup}_{t\leq\tau\leq T} E(H_\tau^0|F_t) = 0\,,$$

we have that $\forall t \geq 0$, $H_t^0 = U_t^0$, hence $\tau_{\min}^0 = 0$. We can conclude that $\operatorname{essinf}_{K\geq 0}(\tau_{\min}^K) = 0$.

(c) From (a) we have that for $K \geq K_0$, $\tau_{\min}^K \leq \tau_{\min}^{K_0}$ P-a.s. If we take $K_0 = 0$, then for $K \geq 0$, $\tau_{\min}^K \leq \tau_{\min}^0$ P-a.s. Using (b) we have that $\tau_{\min}^0 = 0$ so $\tau_{\min}^K = 0$ for $K \geq 0$.

In the discrete time we can prove by a counterexample that the optimal stopping time is not always decreasing. Let's take a one-step binomial tree, with $r = 0.2$, $u = 0.6$, $d = -0.6$, $S_0^1 = 180$, $S_0^0 = 1$, then $p_u = \frac{2}{3}$ and $p_d = \frac{1}{3}$. Then $\tau_{\min}^0 = 0, \tau_{\min}^{10} = 0, \tau_{\min}^{100} = 1, \tau_{\min}^{200} = 1$ and $\tau_{\min}^K = 0$. so we can see that the optimal stopping time is not decreasing.

[1]For more information, see the paper of Nicole Elkaroui and Ioannis Karatzas, *The optimal stopping problem for a general American put option.*

[2]$u_t(K)$ is the value of the American put option at time $t \in [0, T]$, $u_t(K) = esssup_{t\leq\tau\leq T} E_t\left[\frac{(K-S_\tau)_+}{(1+r)^{\tau-t}}\right]$.