Introduction to Mathematical Finance

Exercise sheet 10

Exercise 10.1 Let H be an adapted process in $\mathcal{L}^1(\Omega, \mathcal{F}, Q)$, and define for $\tau \in \mathcal{T}$

$$\mathcal{T}_\tau := \{\sigma \in \mathcal{T} \mid \sigma \geq \tau\}.$$

Show that the *Snell envelope* U^Q of H satisfies $Q-$a.s.

$$U^Q_\tau = \operatorname{ess\,sup}_{\sigma \in \mathcal{T}_\tau} E_Q[H_\sigma \mid \mathcal{F}_\tau],$$

and that the essential supremum is attained for

$$\sigma^{(\tau)}_{\min} := \min\{t \geq \tau \mid H_t = U^Q_t\}.$$

Exercise 10.2

(a) Show that for $Q_1 \approx Q_2$, their pasting in $\sigma \in \mathcal{T}$ is equivalent to Q_1 and satisfies

$$\frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_\sigma},$$

where Z is the density process of Q_2 with respect to Q_1.

(b) Try to find a independent proof of the statement : For $Q_1 \approx Q_2$, let $\tilde{Q}$ be their pasting in $\sigma \in \mathcal{T}$. Then for all stopping times τ and $\mathcal{F}_T$ mesureable $Y \geq 0$,

$$E_{\tilde{Q}}[Y \mid \mathcal{F}_\tau] = E_{Q_1}[E_{Q_2}[Y \mid \mathcal{F}_{\sigma \vee \tau}] | \mathcal{F}_\tau].$$

Exercise 10.3 For a twice differential utility function $U : [0, \infty) \to \mathbb{R}$, the so-called *relative risk aversion* is given by

$$-\frac{xU''(x)}{U'(x)}.$$

(a) Characterize all utility functions $U = U^\gamma$ with constant relative risk aversion equal to γ. Normalize the functions so that $U^\gamma(1) = 0$ and $(U^\gamma)'(1) = 1$.

(b) Verify that $\lim_{\gamma \to 1} U^\gamma(x) = U^1(x)$ for all x.

(c) For a differentiable function $f : [0, \infty) \to [0, \infty)$, the *elasticity* of f is defined as

$$\frac{xf'(x)}{f(x)}.$$

Show that with $U^\gamma(0) = 0$ instead of the normalization above, utility functions with constant relative risk aversion $\gamma \neq 1$ also have constant elasticity.