IMF Serie 10

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Exercise 10.1 Let H be an adapted process in $\mathcal{L}^1(\Omega, \mathcal{F}, Q)$, and define for $\tau \in \mathcal{T}$

$$\mathcal{T}_\tau := \{\sigma \in \mathcal{T} \mid \sigma \geq \tau\}.$$

Show that the *Snell envelope* U^Q of H satisfies Q–a.s.

$$U^Q_\tau = \operatorname{ess\,sup}_{\sigma \in \mathcal{T}_\tau} E_Q[H_\sigma \mid \mathcal{F}_\tau],$$

and that the essential supremum is attained for

$$\sigma^{(\tau)}_{\min} := \min\{t \geq \tau \mid H_t = U^Q_t\}.$$

<u>Proof</u>: <u>Claim 1</u>: $U^Q_\tau = \operatorname{ess-sup}_{\sigma \in \mathcal{T}_\tau} \mathbb{E}_Q(H_\sigma \mid \mathcal{F}_\tau)$

<u>Proof 1</u>: $\rightsquigarrow$ we already know from Thm 6.18 that $\forall t$:

$$U^Q_t = \operatorname{ess-sup}_{t \leq \sigma \leq T} \mathbb{E}_Q(H_\sigma \mid \mathcal{F}_t) \qquad (1)$$

$$\rightsquigarrow U^Q_\tau = \sum_{t=0}^T U^Q_t \mathbb{1}_{\{t=\tau\}} \overset{(1)}{=} \sum_{t=0}^T \operatorname{ess-sup}_{t\leq\sigma\leq T}\left(\mathbb{E}_Q(H_\sigma \mid \mathcal{F}_t)\right) \mathbb{1}_{\{t=\tau\}} =$$

$$= \sum_{t=0}^T \operatorname{ess-sup}_{t\leq\sigma\leq T}\left(\mathbb{E}_Q(H_\sigma \mid \mathcal{F}_t) \cdot \mathbb{1}_{\{t=\tau\}}\right) \mathbb{1}_{\{t=\tau\}} \overset{\text{Claim 1.1}}{=} \sum_{t=0}^T \operatorname{ess-sup}_{t\leq\sigma\leq T}\left(\mathbb{E}_Q(H_\sigma \mid \mathcal{F}_\tau) \mathbb{1}_{\{t=\tau\}}\right) \mathbb{1}_{\{t=\tau\}}$$

$$= \sum_{t=0}^T \operatorname{ess-sup}_{\tau\leq\sigma\leq T}\left(\mathbb{E}_Q(H_\sigma \mid \mathcal{F}_\tau)\right) \mathbb{1}_{\{t=\tau\}} = \operatorname{ess-sup}_{\tau\leq\sigma\leq T} \mathbb{E}_Q(H_\sigma \mid \mathcal{F}_\tau) \underbrace{\sum_{t=0}^T \mathbb{1}_{\{t=\tau\}}}_{=1} \quad Q\text{-a.s.} \quad \square \text{ (Claim 1)}$$

<u>Claim 1.1</u>: for $Y \in \mathcal{L}^1(\Omega, \mathcal{F}, Q)$: $\mathbb{E}_Q(Y \mid \mathcal{F}_t) \cdot \mathbb{1}_{\{\tau=t\}} = \mathbb{E}_Q(Y \mid \mathcal{F}_\tau) \cdot \mathbb{1}_{\{\tau=t\}}$

<u>Proof 1.1</u>: $\rightsquigarrow$ $\{\tau=t\} \in \mathcal{F}_t$ since τ is a stopping time and $\{\tau=t\} \in \mathcal{F}_\tau = \{A \in \mathcal{F} \mid \forall t: A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$

$$\Rightarrow \mathbb{E}_Q(Y \mid \mathcal{F}_t)\mathbb{1}_{\{\tau=t\}} = \mathbb{E}_Q(Y \mathbb{1}_{\{\tau=t\}} \mid \mathcal{F}_t) =: Z_1 \ \ \& \ \ \mathbb{E}_Q(Y \mid \mathcal{F}_\tau)\mathbb{1}_{\{\tau=t\}} = \mathbb{E}_Q(Y \cdot \mathbb{1}_{\{\tau=t\}} \mid \mathcal{F}_\tau) \qquad (1)$$

$\rightsquigarrow$ Z_1 is $\mathcal{F}_t$-mbl (by def.) and $Z_1 = Z_1 \mathbb{1}_{\{\tau=t\}}$ $\Rightarrow \forall B \in \mathcal{B}(\mathbb{R})$: $Z_1^{-1}(B) = (Z_1 \cdot \mathbb{1}_{\{\tau=t\}})^{-1}(B) = \underbrace{\overbrace{Z_1^{-1}(B)}^{\in\mathcal{F}_t} \cap \overbrace{\{\tau=t\}}^{\in\mathcal{F}_t}}_{\in \mathcal{F}_\tau \cap \mathcal{F}_t} + \underbrace{\{\tau\neq t\}}_{\in\mathcal{F}_\tau} \cdot \mathbb{1}_{\{0\in B\}} \in$

$\in \mathcal{F}_\tau \Rightarrow Z_1$ is $\mathcal{F}_\tau$-mbl $\quad (2)$

$$\rightsquigarrow \forall A \in \mathcal{F}_\tau: \ \mathbb{E}_Q(Z_1 \cdot \mathbb{1}_A) \overset{(1)}{=} \mathbb{E}_Q\Big(\mathbb{E}_Q(Y \mid \mathcal{F}_t) \cdot \overbrace{\mathbb{1}_{\{\tau=t\}} \mathbb{1}_A}^{=\mathbb{1}_{\{\tau=t\}\cap A} \in \mathcal{F}_t\text{, since } A \in \mathcal{F}_\tau}\Big) = \mathbb{E}_Q\big(\mathbb{E}_Q(Y \mathbb{1}_{\{\tau=t\}} \mathbb{1}_A \mid \mathcal{F}_t)\big) = \mathbb{E}_Q(Y \mathbb{1}_{\{\tau=t\}} \mathbb{1}_A) \quad (3)$$

$$\overset{(2),(3)}{\Rightarrow} \text{(by def. of the cond. expectation): } \mathbb{E}_Q(Y \mathbb{1}_{\{\tau=t\}} \mid \mathcal{F}_\tau) \underset{Q\text{-a.s.}}{\overset{!}{=}} Z_1 \overset{\text{def}}{=} \mathbb{E}_Q(Y \cdot \mathbb{1}_{\{\tau=t\}} \mid \mathcal{F}_t) \quad \square \text{ (Claim 1.1)}$$

<u>Claim 2</u>: $\sigma^{(\tau)}_{\min}$ fulfils $\mathbb{E}_Q(H_{\sigma^{(\tau)}_{\min}} \mid \mathcal{F}_\tau) = \operatorname{ess-sup}_{\sigma \in \mathcal{T}_\tau} \mathbb{E}_Q(H_\sigma \mid \mathcal{F}_\tau)$

<u>Proof 2</u>: $\rightsquigarrow$ we have $\sigma^{(\tau)}_{\min} \cdot \mathbb{1}_{\{\tau=t\}} = \sigma^{(t)}_{\min} \mathbb{1}_{\{\tau=t\}}$ and by Thm 6.18: $\operatorname{ess-sup}_{t\leq\sigma\leq T} \mathbb{E}_Q(H_\sigma \mid \mathcal{F}_t) = \mathbb{E}_Q(H_{\sigma^{(t)}_{\min}} \mid \mathcal{F}_t) \qquad (4)$

$$\Rightarrow \operatorname{ess-sup}_{\tau\leq\sigma\leq T} \mathbb{E}_Q(H_\sigma \mid \mathcal{F}_\tau) \overset{\text{as in Claim 1}}{=} \sum_{t=0}^T \operatorname{ess-sup}_{t\leq\sigma\leq T}\left(\mathbb{E}_Q(H_\sigma \mid \mathcal{F}_t)\right) \mathbb{1}_{\{\tau=t\}} \overset{(4)}{=} \sum_{t=0}^T \mathbb{E}_Q(H_{\sigma^{(t)}_{\min}} \mid \mathcal{F}_t) \cdot \underbrace{\mathbb{1}_{\{\tau=t\}}}_{\mathcal{F}_t\text{-mbl, since } \tau \text{ stopping time}} = \sum_{t=0}^T \mathbb{E}_Q(\underbrace{H_{\sigma^{(t)}_{\min}} \mathbb{1}_{\{\tau=t\}}}_{=H_{\sigma^{(\tau)}_{\min}} \mathbb{1}_{\{\tau=t\}}\text{, by (4)}} \mid \mathcal{F}_t) =$$

$$= \sum_{t=0}^T \mathbb{E}_Q(H_{\sigma^{(\tau)}_{\min}} \mid \mathcal{F}_t) \mathbb{1}_{\{\tau=t\}} \overset{\text{Claim 1.1}}{=} \sum_{t=0}^T \mathbb{E}_Q(H_{\sigma^{(\tau)}_{\min}} \mid \mathcal{F}_\tau) \mathbb{1}_{\{\tau=t\}} = \mathbb{E}_Q(H_{\sigma^{(\tau)}_{\min}} \mid \mathcal{F}_\tau) \underbrace{\sum_{t=0}^T \mathbb{1}_{\{\tau=t\}}}_{=1} \quad \square \text{ (Claim 2)} \ \blacksquare$$

Exercise 10.2

(a) Show that for $Q_1 \approx Q_2$, their pasting in $\sigma \in \mathcal{T}$ is equivalent to Q_1 and satisfies

$$\frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_\sigma},$$

where Z is the density process of Q_2 with respect to Q_1.

(b) Try to find a independent proof of the statement : For $Q_1 \approx Q_2$, let $\tilde{Q}$ be their pasting in $\sigma \in \mathcal{T}$. Then for all stopping times τ and $\mathcal{F}_T$ mesureable $Y \geq 0$,

$$E_{\tilde{Q}}[Y \mid \mathcal{F}_\tau] = E_{Q_1}[E_{Q_2}[Y \mid \mathcal{F}_{\sigma\vee\tau}] \mid \mathcal{F}_\tau].$$

a.) <u>Proof</u>: $\rightarrow$ $\tilde{Q}$ is given by: (see Def 6.39) $\forall A \in \mathcal{F}_T$:

$$\tilde{Q}(A) = \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(\mathbb{1}_A \mid \mathcal{F}_\sigma))$$

$$\rightsquigarrow \text{if } Q_1(A) = 0 \overset{Q_1\sim Q_2}{\Rightarrow} Q_2(A) = 0 \Rightarrow \tilde{Q}(A) = \mathbb{E}_{Q_1}(\underbrace{\mathbb{E}_{Q_2}(\overbrace{\mathbb{1}_A}^{=0,\ Q_2\text{-a.s.}} \mid \mathcal{F}_\sigma)}_{=0,\ Q_2\text{-a.s. } \&\ Q_1\text{-a.s.}}) = 0 \quad (1)$$

$\rightsquigarrow$ if $Q_1(A) > 0 \Rightarrow Q_2(A) > 0 \Rightarrow \mathbb{E}_{Q_2}(\overbrace{\mathbb{1}_A}^{\geq 0} \mid \mathcal{F}_\sigma) \geq 0$ and $\exists\, \Omega_1 \subset \Omega$ s.t. $Q_2(\Omega_1) > 0$ with $\mathbb{E}_{Q_2}(\mathbb{1}_A \mid \mathcal{F}_\sigma)\mathbb{1}_{\Omega_1} > 0$ (<u>Proof</u>: if not then $0 = \mathbb{E}_{Q_2}(\overbrace{\mathbb{E}_{Q_2}(\mathbb{1}_A \mid \mathcal{F}_\sigma)}^{=0,\ Q_1\text{-a.s.}}) = \mathbb{E}_{Q_2}(\mathbb{1}_A) = Q_2(A) > 0$ ↯)

$$\overset{Q_1\sim Q_2}{\Rightarrow} Q_1(\Omega_1) > 0 \Rightarrow \tilde{Q}(A) = \mathbb{E}_{Q_1}(\underbrace{\mathbb{E}_{Q_2}(\mathbb{1}_A \mid \mathcal{F}_\sigma)}_{\geq 0 \text{ and } >0 \text{ on } \Omega_1 \text{ with } Q_1(\Omega_1)>0}) > 0 \quad (2) \qquad \rightsquigarrow (1) \,\&\, (2) \Rightarrow \tilde{Q} \sim Q_1 \sim Q_2$$

$\rightsquigarrow$ $\frac{d\tilde{Q}}{dQ_1}$ is by def. the Q_1-a.s. unique RV. $S > 0$ s.t. $\forall A \in \mathcal{F}$: $\tilde{Q}(A) = \int_A S \, dQ_1$, (3) $\Rightarrow$ enough to show that $\frac{Z_T}{Z_\sigma}$ fulfils (3)

$\rightsquigarrow$ since $Z = \frac{dQ_2}{dQ_1}$ with $Q_1 \sim Q_2$ we have $Z > 0$ Q_1-a.s. $\Rightarrow$ also $Z_\sigma = \mathbb{E}_{Q_1}(\frac{dQ_2}{dQ_1} \mid \mathcal{F}_\sigma) > 0$ Q_1-a.s

$$\rightsquigarrow \text{let } A \in \mathcal{F} \rightsquigarrow \int_A \frac{Z_T}{Z_\sigma} dQ_1 = \mathbb{E}_{Q_1}\left(\mathbb{1}_A \cdot \frac{Z_T}{Z_\sigma}\right) = \mathbb{E}_{Q_1}\left(\mathbb{E}_{Q_1}\left(\mathbb{1}_A \frac{Z_T}{Z_\sigma} \,\middle|\, \mathcal{F}_\sigma\right)\right) \underset{Z_\tau = \mathbb{E}_{Q_1}(\frac{dQ_2}{dQ_1} \mid \mathcal{F}_\tau)}{\overset{\text{Bayes formula}}{=}} \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(\mathbb{1}_A \mid \mathcal{F}_\sigma)) = \tilde{Q}(A)$$

$$\Rightarrow \frac{Z_T}{Z_\sigma} \text{ fulfils (3)} \Rightarrow \frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_\sigma} \quad Q_1\text{-a.s} \quad \blacksquare$$

b.) <u>Proof</u>: $\rightarrow$ RHS is by def $\mathcal{F}_\tau$-mbl $\Rightarrow$ (by def. of cond. expectation) enough to show: $\forall A \in \mathcal{F}_\tau$: $\mathbb{E}_{\tilde{Q}}(Y \cdot \mathbb{1}_A) = \mathbb{E}_{\tilde{Q}}(\text{RHS} \cdot \mathbb{1}_A) \quad (6)$

$$\rightsquigarrow \text{on } \{\tau \leq \sigma\}: \text{ let } A \in \mathcal{F}_\tau \rightsquigarrow \mathbb{E}_{\tilde{Q}}(Y \cdot \underbrace{\mathbb{1}_A \cdot \mathbb{1}_{\{\tau\leq\sigma\}}}_{\mathcal{F}_\tau \cap \mathcal{F}_\sigma\text{-mbl, by Lem 6.36}}) \overset{\text{def of } \tilde{Q}}{=} \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \underbrace{\mathbb{1}_A \mathbb{1}_{\{\tau\leq\sigma\}}}_{\text{i.e. } \mathcal{F}_\sigma\text{-mbl}} \mid \mathcal{F}_\sigma)) = \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_\sigma) \mathbb{1}_{\{\tau\leq\sigma\}} \mathbb{1}_A) =$$

$$= \mathbb{E}_{Q_1}\Big(\mathbb{E}_{Q_1}\big(\underbrace{\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_\sigma) \cdot \overbrace{\mathbb{1}_{\{\tau\leq\sigma\}}}^{\mathcal{F}_\tau\text{-mbl}}}_{=\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_{\sigma\vee\tau})\mathbb{1}_{\{\tau\leq\sigma\}}\text{, using Claim 1.1 of Ex 10.1 (twice)}} \mathbb{1}_A \mid \mathcal{F}_\tau\big)\Big) = \mathbb{E}_{Q_1}\Big(\underbrace{\mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_{\sigma\vee\tau}) \mid \mathcal{F}_\tau) \ \mathbb{1}_{\{\tau\leq\sigma\}} \mathbb{1}_A}_{\mathcal{F}_\sigma\text{-mbl} \Rightarrow \mathbb{E}_{Q_1}(\cdot) = \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(\cdot \mid \mathcal{F}_\sigma)) = \mathbb{E}_{\tilde{Q}}(\cdot)}\Big) = \mathbb{E}_{\tilde{Q}}\big(\underbrace{\mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_{\sigma\vee\tau}) \mid \mathcal{F}_\tau)}_{=\text{RHS}} \mathbb{1}_{\{\tau\leq\sigma\}} \mathbb{1}_A\big) \quad (4)$$

$$\rightsquigarrow \text{on } \{\tau > \sigma\}: \text{ let } A \in \mathcal{F}_\tau \text{ as above} \rightsquigarrow \mathbb{E}_{\tilde{Q}}(Y \cdot \mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}}) \overset{\text{def. of } \tilde{Q}}{=} \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}} \mid \mathcal{F}_\sigma)) =$$

$$= \mathbb{E}_{Q_1}\Big(\underbrace{\mathbb{E}_{Q_2}(\mathbb{E}_{Q_2}(Y \overbrace{\mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}}}^{\mathcal{F}_\tau\text{-mbl}} \mid \mathcal{F}_\tau) \mid \mathcal{F}_\sigma)}\Big) \overset{\text{def of } \tilde{Q}}{=} \mathbb{E}_{\tilde{Q}}\big(\overbrace{\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_\tau) \mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}}}^{\mathcal{F}_\tau\text{-mbl}}\big)$$

$= \mathbb{E}_{Q_2}(\mathbb{E}_{Q_2}(Y \mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}} \mid \mathcal{F}_\tau) \mid \mathcal{F}_\sigma)$, since this is $\mathcal{F}_\sigma$-mbl & $\forall B \in \mathcal{F}_\sigma$: $\mathbb{E}_{Q_2}(\cdot\ \mathbb{1}_B) = \mathbb{E}_{Q_2}(Y \mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}} \mathbb{1}_B)$ — $\mathbb{1}_{\{\tau>\sigma\}}$ & $\mathbb{1}_B \mathbb{1}_{\{\tau>\sigma\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-mbl, by Lem 6.36 $\Rightarrow$ can be pulled in and out of the cond. expectations.

$$= \mathbb{E}_{\tilde{Q}}\Big(\mathbb{E}_{Q_1}\big(\underbrace{\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_\tau) \cdot \overbrace{\mathbb{1}_{\{\tau>\sigma\}}}^{\mathcal{F}_\tau\text{-mbl}}}_{=\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_{\tau\vee\sigma})\mathbb{1}_{\{\tau>\sigma\}}\text{, using Claim 1.1 of Ex 10.1}} \mathbb{1}_A \mid \mathcal{F}_\tau\big)\Big) =$$

$$= \mathbb{E}_{\tilde{Q}}\big(\underbrace{\mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_{\tau\vee\sigma}) \mid \mathcal{F}_\tau)}_{=\text{RHS}} \mathbb{1}_A \mathbb{1}_{\{\tau>\sigma\}}\big) \quad (5)$$

$\rightsquigarrow$ with (4) & (5) $\Rightarrow \forall A \in \mathcal{F}_\tau$: $\mathbb{E}_{\tilde{Q}}(Y \mathbb{1}_A) = \mathbb{E}_{\tilde{Q}}(Y \mathbb{1}_A \mathbb{1}_{\{\tau \le \sigma\}}) + \mathbb{E}_{\tilde{Q}}(Y \mathbb{1}_A \mathbb{1}_{\{\tau > \sigma\}}) \overset{(4)\&(5)}{=}$

$= \mathbb{E}_{\tilde{Q}}(RHS \cdot \mathbb{1}_A \mathbb{1}_{\{\tau \le \sigma\}}) + \mathbb{E}_{\tilde{Q}}(RHS\, \mathbb{1}_A \mathbb{1}_{\{\tau > \sigma\}}) = \mathbb{E}_{\tilde{Q}}(RHS\, \mathbb{1}_A)$

$\Rightarrow$ (6) holds true $\Rightarrow$ $\mathbb{E}_{\tilde{Q}}(Y \mid \mathcal{F}_\tau) = RHS = \mathbb{E}_{Q_1}(\mathbb{E}_{Q_2}(Y \mid \mathcal{F}_{\tau \vee \sigma}) \mid \mathcal{F}_\tau)$ $\blacksquare$

Exercise 10.3 For a twice differential utility function $U : [0, \infty) \to \mathbb{R}$, the so-called *relative risk aversion* is given by

$$-\frac{xU''(x)}{U'(x)}.$$

(a) Characterize all utility functions $U = U^\gamma$ with constant relative risk aversion equal to γ. Normalize the functions so that $U^\gamma(1) = 0$ and $(U^\gamma)'(1) = 1$.

(b) Verify that $\lim_{\gamma \to 1} U^\gamma(x) = U^1(x)$ for all x.

(c) For a differentiable function $f : [0, \infty) \to [0, \infty)$, the *elasticity* of f is defined as

$$\frac{xf'(x)}{f(x)}.$$

Show that with $U^\gamma(0) = 0$ instead of the normalization above, utility functions with constant relative risk aversion $\gamma \neq 1$ also have constant elasticity.

a) <u>sol</u>: $\rightsquigarrow$ let $U = U^\gamma$ with $-\frac{x\,U''(x)}{U'(x)} = \gamma \quad \forall x \ge 0$

$\rightsquigarrow$ let $g(x) := U'(x) \quad \forall x \ge 0 \Rightarrow -x\, g'(x) = \gamma\, g(x)$

$\Leftrightarrow \frac{g'(x)}{g(x)} = -\frac{\gamma}{x} \rightsquigarrow \int \frac{g'(x)}{g(x)}\,dx = -\int \frac{\gamma}{x}\,dx \Leftrightarrow$ $\quad \frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}\, g'(x)$

$\Leftrightarrow \ln(g(x)) = -\gamma \ln(x) = \ln(x^{-\gamma})$

$\Leftrightarrow g(x) = x^{-\gamma} \Rightarrow$ general solution of the ODE: $g(x) = c_1 x^{-\gamma},\ c_1 \in \mathbb{R}$

$\Rightarrow U'(x) = c_1 x^{-\gamma} \Rightarrow U(x) = c_0 + \int c_1 x^{-\gamma}\,dx, \quad c_0 \in \mathbb{R}$

$\rightsquigarrow$ for $\gamma = 1$: $U(x) = c_0 + \int c_1 x^{-1}\,dx = c_0 + c_1 \ln(x)$

$\rightsquigarrow$ for $\gamma = 0$: $U(x) = c_0 + \int c_1\,dx = c_0 + c_1 x$

$\rightsquigarrow$ for $\gamma \notin \{0,1\}$: $U(x) = c_0 + \int c_1 x^{-\gamma}\,dx = c_0 + c_1 \frac{1}{-\gamma+1} x^{-\gamma+1}$

$\Rightarrow U^\gamma(x) = \begin{cases} c_0 + c_1 \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 1 \\ c_0 + c_1 \ln(x), & \gamma = 1 \end{cases}$

$\rightsquigarrow$ normalizing yields: $1 = (U^\gamma)'(1) = c_1 1^{-\gamma} = c_1 \Leftrightarrow c_1 = 1$ and $0 = U^\gamma(1) = c_0 + \begin{cases} \frac{1}{1-\gamma} 1^{1-\gamma} \\ \ln(1) \end{cases} = c_0 + \begin{cases} \frac{1}{1-\gamma} \\ 0 \end{cases}$

$\Leftrightarrow c_0 = \begin{cases} -\frac{1}{1-\gamma}, & \gamma \neq 1 \\ 0, & \gamma = 1 \end{cases}$

$\Rightarrow$ the normalized utility functions U^γ are given by $U^\gamma(x) = \begin{cases} \frac{1}{\gamma-1} + \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 1 \\ \ln(x), & \gamma = 1 \end{cases}$

b) <u>Proof</u>: $\lim_{\gamma \to 1} U^\gamma(x) = \lim_{\gamma \to 1} \frac{x^{1-\gamma} - 1}{1-\gamma} \overset{1-\gamma =: a}{=} \lim_{a \to 0} \frac{x^a - 1}{a} = \lim_{a \to 0} \frac{\exp(\ln(x^a)) - 1}{a} = \lim_{a \to 0} \frac{\exp(a \ln(x)) - 1}{a} \overset{\text{L'Hospital}}{=} \lim_{a \to 0} \frac{\ln(x) \exp(a \ln(x))}{1} =$

$= \ln(x) = U^1(x)$ $\blacksquare$

c) <u>Proof</u>: $\rightsquigarrow \gamma \neq 1 \Rightarrow U^\gamma(x) = c_0 + \frac{x^{1-\gamma}}{1-\gamma}$ $\rightsquigarrow$ new normalization: $0 = U^\gamma(0) = c_0 \Rightarrow$ normalized utility functions U^γ for $\gamma \neq 1$ are given by: $U^\gamma(x) = \frac{x^{1-\gamma}}{1-\gamma}$

$\Rightarrow \frac{x \cdot (U^\gamma)'(x)}{U^\gamma(x)} = \frac{x \cdot x^{-\gamma}}{\left(\frac{x^{1-\gamma}}{1-\gamma}\right)} = 1 - \gamma = \text{const}$ $\blacksquare$