IMF 10

05.05.17

Maunir S.

10.1) $H \in \mathcal{L}^1(\Omega, \mathcal{F}, Q)$. For $\tau \in \mathcal{T}$, $\mathcal{T}_\tau := \{\sigma \in \mathcal{T} \mid \sigma \geq \tau\}$

Claim: Snell envelope U^Q of H satisfies $U^Q_\tau = \underset{\sigma \in \mathcal{T}_\tau}{\text{ess-sup}}\, E_Q[H_\sigma \mid \mathcal{F}_\tau]$ Q-a.s.

Proof: We proof this by claiming $\underbrace{E_Q[H_\sigma \mid \mathcal{F}_\tau] = E_Q[H_\sigma \mid \mathcal{F}_t] \text{ Q-a.s. on } \{\tau = t\}}_{(*)}$.

With $(*)$ we can simply write,

$$E_Q[H_\sigma \mid \mathcal{F}_\tau] = \sum_{t=0}^{T} E_Q[H_\sigma \mid \mathcal{F}_\tau]\, 1_{\{\tau=t\}} \overset{(*)}{=} \sum_{t=0}^{T} E_Q[H_\sigma \mid \mathcal{F}_t]\, 1_{\{\tau=t\}}$$

Hence,

$$\underset{\sigma \in \mathcal{T}_\tau}{\text{ess-sup}}\, E_Q[H_\sigma \mid \mathcal{F}_\tau] = \sum_{t=0}^{T} \underbrace{\underset{\sigma \in \mathcal{T}_t}{\text{ess-sup}}\, E_Q[H_\sigma \mid \mathcal{F}_t]}_{= U^Q_t \text{ by Thm 6.18}} 1_{\{\tau=t\}} = \sum_{t=0}^{T} U^Q_t 1_{\{\tau=t\}}$$

$$= \sum_{t=0}^{T} U^Q_\tau 1_{\{\tau=t\}} = U^Q_\tau \sum_{t=0}^{T} 1_{\{\tau=t\}}$$

$$= U^Q_\tau.$$

So it remains to show $(*)$: We want $E_Q[H_\sigma \mid \mathcal{F}_\tau]\, 1_{\{\tau=t\}} \overset{Q\text{-a.s.}}{=} E_Q[H_\sigma \mid \mathcal{F}_t]\, 1_{\{\tau=t\}}$,

i.e. for $A \in \mathcal{F}_\tau$: $E_Q[H_\sigma 1_{\{\tau=t\} \cap A}] = E_Q[E[H_\sigma \mid \mathcal{F}_t]\, 1_{\{\tau=t\} \cap A}]$

Since $\tau \in \mathcal{T}$ clearly $\{\tau = t\} \in \mathcal{F}_t, \mathcal{F}_\tau$

Then

$$E_Q[H_\sigma 1_{\{\tau=t\}} 1_A] = E_Q[E_Q[H_\sigma 1_{\{\tau=t\}} 1_A \mid \mathcal{F}_t]]$$

$$= E_Q[E_Q[H_\sigma 1_{\{\tau=t\}} 1_A \mid \mathcal{F}_\tau]]$$

$$= E_Q[E_Q[H_\sigma 1_{\{\tau=t\}} \mid \mathcal{F}_\tau]\, 1_A]$$

$\square$

Claim: $U^Q_\tau = E_Q[H_{\sigma^{(\tau)}_{min}} \mid \mathcal{F}_\tau]$ Q-a.s. for $\sigma^{(\tau)}_{min} := \min\{t \geq \tau \mid H_t = U^Q_t\}$

Proof: Again on $\{\tau = t\}$ we have $U^Q_t = E_Q[H_{\sigma^{(t)}_{min}} \mid \mathcal{F}_t] = \underset{\sigma \in \mathcal{T}_t}{\text{ess-sup}}\, E_Q[H_\sigma \mid \mathcal{F}_t]$ (Thm 6.18)

So

$$U^Q_\tau = \underset{\sigma \in \mathcal{T}_\tau}{\text{ess-sup}}\, E_Q[H_\sigma \mid \mathcal{F}_\tau] = \sum_{t=0}^{T} \underbrace{\underset{\sigma \in \mathcal{T}_t}{\text{ess-sup}}\, E_Q[H_\sigma \mid \mathcal{F}_t]}_{= U^Q_t = E_Q[H_{\sigma^{(t)}_{min}} \mid \mathcal{F}_t]} 1_{\{\tau=t\}}$$

$$= \sum_{t=0}^{T} U^Q_t 1_{\{\tau=t\}}$$

$$= \sum_{t=0}^{T} E_Q[H_{\sigma^{(t)}_{min}} \mid \mathcal{F}_t]\, 1_{\{\tau=t\}} = \sum_{t=0}^{T} E_Q[H_{\sigma^{(\tau)}_{min}} \mid \mathcal{F}_\tau]\, 1_{\{\tau=t\}} = E_Q[H_{\sigma^{(\tau)}_{min}} \mid \mathcal{F}_\tau]$$

backside $\Longrightarrow$ $\square$

$(*)$ backside: for $\gamma = 1$: $v(x) = \frac{c_1}{x}$ $\Rightarrow$ $U^1(x) = \int v(x)\,dx = c_1 \log(x) \overset{\text{normalize}}{=} \log(x)$

10.2) a) Let $\tilde{Q}$ be the pasting of Q_1 and $Q_2\ (\approx Q_1)$ along $\sigma \in \mathcal{T}$.

Let $Z_T = \frac{dQ_2}{dQ_1}$ be the density process of Q_2 wrt. Q_1, $Z_t = E[Z \mid \mathcal{F}_t]$.

Claim: $\tilde{Q} \approx Q_1$ and $\frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_\sigma}$

Proof: Since $Z_T = E\left[\frac{dQ_2}{dQ_1} \mid \mathcal{F}_T\right] = \frac{dQ_2}{dQ_1} > 0 \Rightarrow$ equivalence of $\tilde{Q}$ and Q_1

Furthermore for $Y \geq 0$:

$$E_{\tilde{Q}}[Y] \underset{\substack{\uparrow \\ \text{Remark} \\ \text{in lecture}}}{=} E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_\sigma]\right]$$

Bayes' formula:

$$E_{Q_2}[\underset{\mathcal{F}_T\text{-mbl}}{Y} \mid \mathcal{F}_\sigma] = \frac{1}{Z_\sigma} E_{Q_1}[Z_T Y \mid \mathcal{F}_\sigma]$$

$$= E_{Q_1}\left[E_{Q_1}[Z_T Y \mid \mathcal{F}_\sigma] \frac{1}{Z_\sigma}\right]$$

$$= E_{Q_1}\left[E_{Q_1}\left[\frac{Z_T}{Z_\sigma} Y \mid \mathcal{F}_\sigma\right]\right] = E_{Q_1}\left[\frac{Z_T}{Z_\sigma} Y\right]$$

$$\Rightarrow \frac{d\tilde{Q}}{dQ_1} = \frac{Z_T}{Z_\sigma}.$$

$\square$

b) Claim: For $\tau \in \mathcal{T}$, $Y \geq 0$ $\mathcal{F}_T$-mbl, $E_{\tilde{Q}}[Y \mid \mathcal{F}_\tau] = E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_{\sigma \vee \tau}] \mid \mathcal{F}_\tau\right]$

Proof: Take some φ $\mathcal{F}_\tau$-mbl. Then

- $$E_{\tilde{Q}}[Y \underbrace{\varphi \mathbb{1}_{\{\tau \leq \sigma\}}}_{\in \mathcal{F}_\tau \cap \mathcal{F}_\sigma\text{-mbl}}] = E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_\sigma]\, \varphi\, \mathbb{1}_{\{\tau \leq \sigma\}}\right]$$

$$= E_{Q_1}\left[\underbrace{E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_\sigma] \mid \mathcal{F}_\tau\right] \varphi \mathbb{1}_{\{\tau \leq \sigma\}}}_{\in \mathcal{F}_\sigma}\right]$$

on $\mathcal{F}_\sigma$: $\tilde{Q} = Q_1$

$$\underset{a)}{=} E_{\tilde{Q}}\left[E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_\sigma] \mid \mathcal{F}_\tau\right] \varphi \mathbb{1}_{\{\tau \leq \sigma\}}\right]$$

- $$E_{\tilde{Q}}[Y \varphi \mathbb{1}_{\{\tau > \sigma\}}] = E_{Q_1}\left[E_{Q_2}\left[E_{Q_2}[Y \mid \mathcal{F}_\tau]\, \varphi \mid \mathcal{F}_\sigma\right] \mathbb{1}_{\{\tau > \sigma\}}\right]$$

$$= E_{\tilde{Q}}\left[E_{Q_2}[Y \mid \mathcal{F}_\tau]\, \varphi\, \mathbb{1}_{\{\tau > \sigma\}}\right]$$

$$\Rightarrow E[Y \mid \mathcal{F}_\tau] = \mathbb{1}_{\{\tau \leq \sigma\}} E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_\sigma] \mid \mathcal{F}_\tau\right] + \mathbb{1}_{\{\tau > \sigma\}} E_{Q_2}[Y \mid \mathcal{F}_\tau]$$

$$= E_{Q_1}\left[E_{Q_2}[Y \mid \mathcal{F}_{\sigma \vee \tau}] \mid \mathcal{F}_\tau\right].$$

$\square$

10.3) a) $U'' - \frac{\gamma}{x} U' = 0 \overset{v := u'}{\Longrightarrow} v' - \frac{\gamma}{x} v = 0 \Rightarrow v(x) = c_1 e^{\int(-\frac{\gamma}{x})dx} = c_1 x^{-\gamma}$

$\Rightarrow U = \int v\, dx = \frac{c_1 x^{1-\gamma}}{1-\gamma} + C_2$. $U(1) = 0 \Rightarrow C_2 = -\frac{c_1}{1-\gamma}$, $U'(1) = 1 \Rightarrow C_1 = 1$

$\Rightarrow \boxed{U^\gamma(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}}, \gamma \neq 1$

b) $\lim_{\gamma \to 1} U^\gamma(x) = \lim_{\gamma \to 1} \frac{x^{1-\gamma} - 1}{1-\gamma} \overset{q}{=} \lim_{q \to 0} \frac{e^{\log(x) q} - 1}{q} = \lim_{q \to 0} \frac{\log(x) e^{\log(x) q}}{1} = \log(x) = U^1(x)$ ✱

$\Rightarrow \frac{x\, U'(x)}{U(x)} = \frac{x\, c_1 x^{-\gamma}}{c_1 x^{1-\gamma}} (1-\gamma) = 1-\gamma = \text{const}$

c) $U^\gamma(x) = \frac{c_1 x^{1-\gamma}}{1-\gamma} + C_2$, $U^\gamma(0) = 0 \Rightarrow C_2 = 0 \Rightarrow U^\gamma(x) = \frac{c_1 x^{1-\gamma}}{1-\gamma}$

elasticity