Introduction to Mathematical Finance

Solution sheet 10

Solution 10.1 Fix γ and write $U := U^\gamma$.

(a) Since the ODE

$$\frac{U''(x)}{U'(x)} = -\frac{\gamma}{x}$$

is separable, we find U' from

$$\ln U' = \int \frac{1}{U'} dU' = -\int \frac{\gamma}{x} dx = -\gamma \ln x + C.$$

From $U'(1) = 1$, we obtain $C = 0$. Hence,

$$U'(x) = x^{-\gamma}.$$

Integrate again to get

$$U = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} + D, & \text{if } \gamma \neq 1, \\ \ln x + D, & \text{if } \gamma = 1. \end{cases}$$

The condition $U(1) = 0$ gives $D = -1/(1-\gamma)$ and $D = 0$ for the two cases, respectively. The utility function $U = U^\gamma$ is therefore given by

$$U^\gamma(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma}, & \text{if } \gamma \neq 1, \\ \ln x, & \text{if } \gamma = 1. \end{cases}$$

If we also want U to be concave, we have to impose $\gamma \geq 0$.

(b) We employ L'Hopital's rule:

$$\lim_{\gamma \to 1} \frac{x^{1-\gamma} - 1}{1-\gamma} = \lim_{\gamma \to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}\gamma}(x^{1-\gamma} - 1)}{\frac{\mathrm{d}}{\mathrm{d}\gamma}(1-\gamma)} = \lim_{\gamma \to 1} \frac{-x^{1-\gamma} \ln x}{-1} = \ln x,$$

which is what we wanted to show.

(c) If we normalize the utility functions found in (a) such that $D = 0$, we obtain

$$U^\gamma(x) = \frac{x^{1-\gamma}}{1-\gamma}.$$

Plugging this into the expression for elasticity gives

$$\frac{x^{-\gamma}}{\frac{x^{1-\gamma}}{1-\gamma}} x = 1 - \gamma.$$

Hence, U^γ has constant elasticity $1 - \gamma$.

Remark: This type of utility function is called *isoelastic*. Since an additive constant only shifts the 'utility' by a fixed amount and does not change the behavior under, for example, maximization, this name is also used for the utility functions found in (a).