Introduction to mathematical finance

Exercise sheet 11

Exercise 11.1 For probability measures $Q \ll P$ the *(relative) entropy* of Q with respect to P is defined as

$$H(Q|P) = E_P\left[\tfrac{\mathrm{d}Q}{\mathrm{d}P}\ln\tfrac{\mathrm{d}Q}{\mathrm{d}P}\right] = E_Q\left[\ln\tfrac{\mathrm{d}Q}{\mathrm{d}P}\right].$$

In this problem, we consider the trinomial market with $m = r = 0$, $u = -d$, $\pi^1 = 1$ and $p^i = P[S_1^1 = 1 + i]$.

1. Find the measure Q^* minimizing the relative entropy $H(Q|P)$ over all equivalent martingale measures Q.

2. Find the strategy ξ^* maximizing expected utility of final wealth, with initial wealth 0 and exponential utility with parameter α, i.e.,

   $$U_w(x) = 1 - e^{-\alpha x} \quad \text{and} \quad U_c(x) = 0.$$

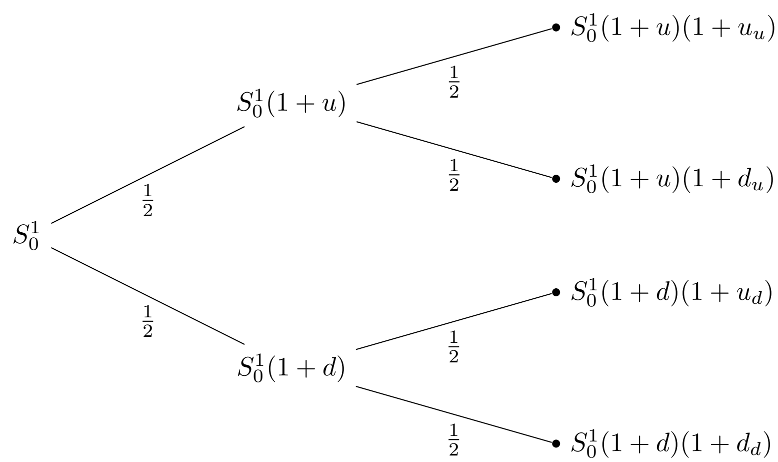
   Verify that

   $$\frac{\mathrm{d}Q^*}{\mathrm{d}P} = \frac{e^{\eta^*\cdot\Delta X_1}}{E[e^{\eta^*\cdot\Delta X_1}]},$$

   with $\eta^* = -\alpha\xi^*$.

Exercise 11.2

Consider the asset given by the following tree:



where the fractions denote probabilities. Let $S_k^0 = (1+r)^k$ and

$$-1 < r \in (d,u) \cap (d_u,u_u) \cap (d_d,u_d) \neq \emptyset,$$

i.e., the market is arbitrage-free.

Strategies here can be identified with vectors in $\mathbb{R}^3$ via $\xi = (\xi_1, \xi_2^u, \xi_2^d)$. Find the optimizer ξ^* to the problem of maximizing (exponential) utility of final wealth:

$$\max_{\xi\in\mathbb{R}^3} E\left[1 - \exp\left(-v_0 - (\xi \bullet X)_2\right)\right].$$

Exercise 11.3 Consider a general arbitrage-free single-period market.Fix x and let $U : [0, \infty) \to \mathbb{R}$ be a concave, increasing (utility) function, continuously differentiable on $(0, \infty)$, such that

$$\sup_{\xi \in \mathcal{A}(x)} E[U(x + \xi \cdot \Delta X_1)] < \infty,$$

with

$$\mathcal{A}(x) = \{\xi \in \mathbb{R}^d | x + \xi \cdot \Delta X_1 \geq 0 \, P\text{-a.s.}\}.$$

Furthermore, assume that the supremum is attained in an interior point ξ^* of $\mathcal{A}(x)$.

1. Show that
$$U'(x + \xi^* \cdot \Delta X_1)|\Delta X_1| \in L^1(P)$$
and the *first order condition*
$$E[U'(X + \xi^* \cdot \Delta X_1)\Delta X_1] = 0.$$
Hint: You may use that
$$y \mapsto \frac{U(y) - U(z)}{y - z}, \quad y \in (0, \infty) \setminus \{z\}$$
is nonincreasing. By optimality, ξ^* is better than $\xi^* + \varepsilon\eta$ for any $\eta \neq 0$ and $0 < \varepsilon \ll 1$; so take the difference of corresponding utilities, divide by ε and look at $\varepsilon \searrow 0$. Exploit the hint to see that this quantity is monotonic in ε.

2. Show that Q given by
$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}P} = \frac{U'(x + \xi^* \cdot \Delta X_1)}{E[U'(x + \xi^* \cdot \Delta X_1)]}$$
is an equivalent martingale measure.