Introduction to mathematical finance

Solution sheet 11

Solution 11.1

1. The parametrization of $\mathbb{P}$ is

$$\left\{ \left(\frac{1-\lambda}{2}, \lambda, \frac{1-\lambda}{2} \right) \middle| \lambda \in (0,1) \right\}.$$

Using this, we can write down the relative entropy given the parameter λ as

$$\begin{aligned} H(Q^\lambda | P) &= \frac{1-\lambda}{2} \left(\ln \frac{1-\lambda}{2p^u} + \ln \frac{1-\lambda}{2p^d} \right) + \lambda \ln \frac{\lambda}{p^m} \\ &= (1-\lambda) \left(\ln(1-\lambda) - \ln \left(2\sqrt{p^u p^d} \right) \right) + \lambda \ln \lambda - \lambda \ln p^m. \end{aligned}$$

Minimizing this expression over $Q^\lambda \in \mathbb{P}$ yields the necessary condition

$$-\left(\ln(1-\lambda^*) - \ln(2\sqrt{p^u p^d}) \right) - 1 + \ln \lambda^* + 1 - \ln p^m = 0.$$

This condition is also sufficient since the second derivative is positive. Solving for λ^*, one obtains

$$\frac{\lambda^*}{1-\lambda^*} = p^m / 2\sqrt{p^u p^d},$$

and from there

$$\lambda^* = \frac{p^m}{2\sqrt{p^u p^d}} \frac{1}{1 + \frac{p^m}{2\sqrt{p^u p^d}}} = \frac{p^m}{p^m + 2\sqrt{p^u p^d}}.$$

Thus,

$$Q^* = Q^{\lambda^*} = \frac{(\sqrt{p^u p^d}, p^m \sqrt{p^u p^d})}{p^m + 2\sqrt{p^u p^d}}.$$

2. First note that maximizing the expected utility is equivalent to minimizing

$$E[e^{-\alpha \xi \cdot \Delta X_1}]$$

over ξ, or

$$E[e^{\eta \cdot \Delta X_1}]$$

over η, i.e., minimizing the moment generating function of ΔX_1.

Write Z for the moment generating function of ΔX_1. We begin by finding the minimizer η^*. The necessary condition

$$\begin{aligned} Z'(\eta^*) &= \frac{\mathrm{d}}{\mathrm{d}\eta} \left(p^u e^{\eta u X_0^1} + p^m + p^d e^{\eta d X_0^1} \right) \Big|_{\eta = \eta^*} \\ &= p^u u X_0^1 e^{\eta^* u X_0^1} + p^d d X_0^1 e^{\eta^* d X_0^1} = 0 \end{aligned}$$

simplifies to

$$\eta^* = \frac{\ln\left(-\frac{p^d d}{p^u u}\right)}{(u-d)X_0^1} = \frac{\ln\left(\frac{p^d}{p^u}\right)}{2uX_0^1}.$$

The second derivative of Z is easily verified to be positive, and therefore η^* is a minimizer. The corresponding strategy is then $\xi^* = -\eta^*/\alpha$.

This gives

$$Z(\eta^*) = p^m + 2\sqrt{p^d p^u},$$

and in turn also the measure

$$\frac{(\sqrt{p^u p^d}, p^m, \sqrt{p^u p^d})}{Z(\eta^*)}.$$

This is precisely what we found in (a).

Solution 11.2 The objective function is given by

$$E[1 - e^{-(v_0 + (\bar{\xi} \bullet X)_2}] = 1 - \frac{1}{4} \sum_{\substack{i \in \{u,d\} \\ j \in \{u_i, d_i\}}} \underbrace{\exp\left(-v_0 - \bar{\xi}_1 \frac{i-r}{1+r} S_0^1 - \bar{\xi}_2^i \frac{(j-r)(1+i)}{(1+r)^2} S_0^1\right)}_{=:D(i,j)}.$$

Begin by finding $\bar{\xi}_2^i$ by differentiating this expression with respect to $\bar{\xi}_2^i$. Setting the derivative to zero yields

$$\frac{(u_i - r)(1+i)}{(1+r)^2} S_0^1 D(i, u_i) + \frac{(d_i - r)(1+i)}{(1+r)^2} S_0^1 D(i, d_i) = 0. \tag{1}$$

Combining the two factors $D(i,j)$, taking logarithms and finally solving for $\bar{\xi}_2^i$ yields

$$\bar{\xi}_2^i = -\frac{(1+r)^2}{(u_i - d_i)(1+i) S_0^1} \ln \frac{r - d_i}{u_i - r}.$$

The corresponding condition for $\bar{\xi}_1$ is given by

$$\sum_{\substack{i \in \{u,d\} \\ j \in \{u_i, d_i\}}} \frac{i-r}{1+r} S_0^1 D(i,j) = 0.$$

Eliminate $D(i, d_i)$ with (1) to obtain

$$\sum_{i \in \{u,d\}} \frac{i-r}{1+r} \left(1 - \frac{u_i - r}{d_i - r}\right) D(i, u_i) = \sum_{i \in \{u,d\}} \frac{i-r}{1+r} \frac{u_i - d_i}{r - d_i} D(i, u_i) = 0.$$

This can be rewritten as

$$\exp\left(-\bar{\xi}_1 \frac{u-d}{1+r} S_0^1\right) = \frac{r-d}{u-r} \frac{u_d - d_d}{r - d_d} \frac{r - d_u}{u_u - d_u}$$
$$\exp\left(-\bar{\xi}_2^d \frac{(u_d - r)(1+d)}{(1+r)^2} S_0^1\right) \exp\left(\bar{\xi}_2^u \frac{(u_u - r)(1+u)}{(1+r)^2} S_0^1\right).$$

Finally, the values for $\bar{\xi}_2^u$ and $\bar{\xi}_2^d$ can be plugged in to solve for $\bar{\xi}_1$:

$$\bar{\xi}_1 \frac{u-d}{1+r} S_0^1 = -\ln\left(\frac{r-d}{u-r} \frac{u_d - d_d}{r - d_d} \frac{r - d_u}{u_u - d_u}\right)$$
$$- \left(\frac{u_d - r}{u_d - d_d} \ln \frac{r - d_d}{u_d - r}\right) + \left(\frac{u_u - r}{u_u - d_u} \ln \frac{r - d_u}{u_u - r}\right).$$

The optimizer is thus given by $\xi^* = (\bar{\xi}_1, \bar{\xi}_2^u, \bar{\xi}_2^d)$.

Solution 11.3

1. Let η be any non-zero vector. Then, by the assumption that ξ^* is an interior point, $\xi^* + \varepsilon\eta \in \mathcal{A}(x)$ for all $0 < \varepsilon \ll 1$. Define

$$\Delta_\varepsilon^\eta = \frac{U(x + (\xi^* + \varepsilon\eta) \cdot \Delta X_1) - U(x + \xi^* \cdot \Delta X_1)}{\varepsilon},$$

for small ε, as above. On $\{\eta \cdot \Delta X_1 = 0\}$, $\Delta_\varepsilon^\eta \equiv 0$, and on $\{\eta \cdot \Delta X_1 \neq 0\}$,

$$\Delta_\varepsilon^\eta = \eta \cdot \Delta X_1 \frac{U(x + (\xi^* + \varepsilon\eta) \cdot \Delta X_1) - U(x + \xi^* \cdot \Delta X_1)}{\varepsilon\eta \cdot \Delta X_1},$$

so Δ_ε^η is monotonically[1] increasing to $\eta \cdot \Delta X_1 U'(x + \xi^* \cdot \Delta X_1)$ as $\varepsilon \searrow 0$. Note that by the hint, $U'(0)$ is well-defined.

From (**??**) we know that $U(x + \xi \cdot \Delta X_1) \in L^1$ for all ξ, and so $\Delta_\varepsilon^\eta \in L^1$. Next note that $\varepsilon \mapsto \Delta_\varepsilon^\eta$ is decreasing; so for $\varepsilon \searrow 0$, $\Delta_\varepsilon^\eta \nearrow$ and we can use monotone convergence. Therefore, by monotone convergence and then optimality of ξ^*,

$$-\infty < E[\Delta_\varepsilon^\eta] \le E[U'(x + \xi^* \cdot \Delta X_1)\eta \cdot \Delta X_1] = \lim_{\varepsilon \searrow 0} E[\Delta_\varepsilon^\eta] \le 0.$$

Therefore, $U'(x + \xi^* \cdot \Delta X_1)\eta \cdot \Delta X_1 \in L^1(P)$. Finally, since η can be chosen arbitrarily, $U'(x + \xi^* \cdot \Delta X_1)\Delta X_1 \in L^1(P)$ and

$$\eta \cdot E[U'(x + \xi^* \cdot \Delta X_1)\Delta X_1] \le 0$$

with $\eta = E[U'(x + \xi^* \cdot \Delta X_1)\Delta X_1]$ implies

$$E[U'(x + \xi^* \cdot \Delta X_1)\Delta X_1] = 0.$$

2. By part (a), $\bar{Q}$ is an EMM if we can show that it is well-defined, i.e., $U'(x + \xi^* \cdot \Delta X_1) \in L^1(P)$. Observe that

$$U'(x + \xi^* \cdot \Delta X_1) = U'(x + \xi^* \cdot \Delta X_1) 1_{\{\xi^* \cdot \Delta X_1 \le -x/2\}} + U'(x + \xi^* \cdot \Delta X_1) 1_{\{\xi^* \cdot \Delta X_1 \ge -x/2\}}.$$

The second term is bounded by $U'(x/2)$ since U' is non-increasing. Again using part (a),

$$\begin{aligned} &E[U'(x + \xi^* \cdot \Delta X_1) 1_{\{\xi^* \cdot \Delta X_1 \le -x/2\}}] \\ &\qquad \le E\left[\frac{-\xi^* \cdot \Delta X_1}{x/2} U'(x + \xi^* \cdot \Delta X_1) 1_{\{\xi^* \cdot \Delta X_1 \le -x/2\}}\right] \\ &\qquad \le \frac{2}{x} E[|\xi^* \cdot \Delta X_1| U'(x + \xi^* \cdot \Delta X_1)] < \infty. \end{aligned}$$

[1] This is easily seen by splitting into two cases depending on the sign of $\eta \cdot \Delta X_1$.