Introduction to mathematical finance

Exercise sheet 12

Exercise 12.1 Consider the arbitrage-free market in T periods with a riskless bond with zero interest rate. Assume that $H \in L^0_+(\mathcal{F}_T)$ is an attainable claim and fix one EMM Q. Let U be an exponential utility function, $\mathcal{A} = \Theta$ be the set of predictable processes, and consider the two functions

$$u(x) = \max_{\xi \in \mathcal{A}} E[U(x + (\xi \bullet X)_T)]$$

and

$$u_H(x) = \max_{\xi \in \mathcal{A}} E[U(x + (\xi \bullet X)_T - H)].$$

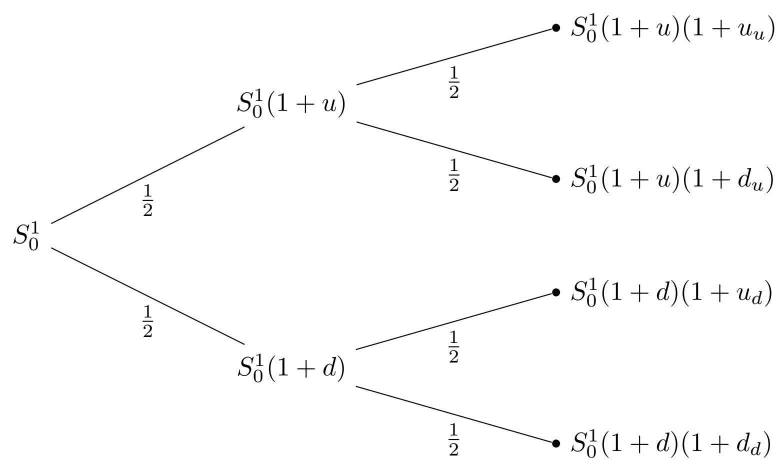
Given a wealth level x, the *utility indifference price* $p_H(x)$ of H is defined as the solution to

$$u(x) = u_H(x + p_H(x)).$$

1. Show that $E_Q[H]$ is the unique solution to the above equation.
2. Can the assumptions on the utility function be generalized?

Exercise 12.2

Recall the market structure from Exercise 11.2:



where the fractions denote probabilities. Let $S^0_k = (1+r)^k$ and

$$-1 < r \in (d, u) \cap (d_u, u_u) \cap (d_d, u_d) \neq \emptyset.$$

Strategies are identified with vectors in $\mathbb{R}^3$ via $\xi = (\xi_1, \xi_2^u, \xi_2^d)$. Find the optimizer ξ^* to the problem

$$\max_{\xi \in \mathbb{R}^3} E\left[1 - \exp\left(-v_0 - (\xi \bullet X)_2\right)\right].$$

using dynamic programming[1].

[1] dynamic programming is what we call the backward recursion.

Exercise 12.3 Consider an individual with endowment (income) $(e_0, \ldots, e_T)$, $e_0 > 0$ and who only invests in the riskless bank account $S_k^0 = (1+r)^k$. Denote by W_k the wealth held in the bank account when leaving time point $k-1$ and $W_0 = v_0$. Then, for any consumption strategy,

$$W_{k+1} = (W_k + e_k - c_k)(1+r).$$

With $\beta \in (0, 1]$, the individual aims to maximize

$$E\left[\sum_{k=0}^{T} \beta^k U(c_k)\right],$$

subject to

$$v_0 + \sum_{k=0}^{T} e^{-rk}(e_k - c_k) = 0,$$

over all (possibly negative) consumption plans, and where $\tilde{\beta} = 1/\beta - 1$ is a so-called *impatience parameter*. For simplicity, we will consider $\tilde{\beta} = r$. Assume that U is described by a quadratic parabola, so that U' is an affine function.[2]

Find the optimal consumption strategy.

[2] We make this assumption for the sake of the exercise, even though a parabola is not increasing on a sufficiently large domain.