# Introduction to mathematical finance

## Solution sheet 12

**Solution 12.1**

1. Let $\xi_H$ be a replicating strategy, i.e., $E_Q[H] + G_T(\xi_H) = H$ $P$-a.s. Then, for any $\xi \in \mathcal{A}$,

$$(\xi \bullet X)_T - H = (\xi \bullet X)_T - (\xi_H \bullet X)_T - E_Q[H] = ((\xi - \xi_H) \bullet X)_T - E_Q[H].$$

Hence,

$$\begin{aligned} u_H(x + E_Q[H]) &= \max_{\xi \in \mathcal{A}} E[U(x + E_Q[H] + (\xi \bullet X)_T - H)] \\ &= \max_{\xi \in \mathcal{A}} E[U(x + ((\xi - \xi_H) \bullet X)_T)] \\ &= \max_{\xi \in \mathcal{A} - \xi_H} E[U(x + (\xi \bullet X)_T)] = u(x), \end{aligned}$$

since $\mathcal{A} = \mathcal{A} - \xi_H$. Thus, $p_H(x) = E_Q[H]$ is a solution. Since $U$ is strictly increasing, with any other choice for $p_H(x)$, we would obtain a strict inequality. Therefore, our solution is unique.

*Remark:* This type of problem is mostly of interest for non-attainable claims.

2. We have only used two properties of $U$. All expressions must be well defined, which means $U$ must be defined on all of $\mathbb{R}$. For uniqueness we also need $U$ to be strictly increasing.

**Solution 12.2** We write $X = X^1$. At the final time, $\mathcal{O}_2(v_2) = 1 - \exp(-v_2)$. By the dynamic programming principle,

$$\mathcal{O}_1(v_1) = \operatorname{ess\,sup}_{\bar{\xi}_2} E[1 - \exp(-v_1 - \bar{\xi}_2 \Delta X_2) | \mathcal{F}_1].$$

Note that this quantity takes two different values. Use $i \in \{u, d\}$ to distighuish between the two possible outcomes in the first step. The first order condition is then

$$\frac{u_i - r}{1 + r} X_1^i \exp\left(-v_1^i - \bar{\xi}_2^i \frac{u_i - r}{1 + r} X_1^i\right) \\ + \frac{d_i - r}{1 + r} X_1^i \exp\left(-v_1^i - \bar{\xi}_2^i \frac{d_i - r}{1 + r} X_1^i\right) = 0.$$

Solving for $\bar{\xi}_2^i$ gives

$$\bar{\xi}_2^i = -\frac{1 + r}{(u_i - d_i) X_1^i} \ln \frac{r - d_i}{u_i - r} = -\frac{(1 + r)^2}{(u_i - d_i)(1 + i) X_0} \ln \frac{r - d_i}{u_i - r},$$

which, as expected, is the same result as found in Exercise 11.2. Thus,

$$\begin{aligned} \mathcal{O}_1(v_1) &= E[1 - \exp(-v_1 - \bar{\xi}_2 \Delta X_2) | \mathcal{F}_1], \\ &= 1 - e^{-v_1} \underbrace{E[\exp(-\bar{\xi}_2 \Delta X_2) | \mathcal{F}_1]}_{=: N_1}. \end{aligned}$$

Proceed with finding $\mathcal{O}_0$:

$$\begin{aligned}\mathcal{O}_0(v_0) &= \operatorname{ess\,sup}_{\bar{\xi}_1} E[\mathcal{O}_1(v_0 + \bar{\xi}_1 \Delta X_1)] \\ &= \operatorname{ess\,sup}_{\bar{\xi}_1} 1 - E[\exp(-v_0 - \bar{\xi}_1 \Delta X_1) N_1] \\ &= \operatorname{ess\,sup}_{\bar{\xi}_1} 1 - \frac{e^{-v_0}}{2} \sum_{i \in \{u,d\}} e^{-\bar{\xi}_1 \Delta X_1^i} N_1^i.\end{aligned}$$

Differentiating with respect to $\bar{\xi}_1$ and setting the derivative to zero, yields

$$e^{-\bar{\xi}_1 \Delta X_1^u} N_1^u + e^{-\bar{\xi}_1 \Delta X_1^d} N_1^d = 0.$$

Solve for $\bar{\xi}_1$ to obtain

$$\bar{\xi}_1 = -\frac{1+r}{(u-d)S_0^1} \ln\left(-\frac{N_1^d}{N_1^u}\right).$$

With the elimination using (1) from Solution 11.2, we can verify that this is indeed the same solution as we found using direct methods.

**Solution 12.3** Suppose $c^*$ is the maximizer. Then, for any $k \le T-1$, $t \le T-k$, $A \in \mathcal{F}_k$ and

$$c^\varepsilon = (c_0^*, \ldots, c_{k-1}^*,\ c_k^* - \varepsilon 1_A,\ c_{k+1}^*, \ldots, c_{k+t-1}^*,\ c_{k+t}^* + (1+r)^t \varepsilon 1_A,\ c_{k+t+1}^*, \ldots, c_T^*),$$

we have that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} E\left[\sum_{k=0}^T \beta^k U(c_k^\varepsilon)\right]\Bigg|_{\varepsilon=0} = \beta^k E\left[-U'(c_k^*)1_A + \beta^t (1+r)^t U'(c_{k+t}^*)1_A\right] = 0.$$

Since this holds for any $A \in \mathcal{F}_k$ and $\tilde{\beta} = r$, we conclude that

$$U'(c_k^*) = E[U'(c_{k+t}^*)|\mathcal{F}_k],$$

and by the parabolic property of $U$, that

$$c_k^* = E[c_{k+t}^*|\mathcal{F}_k].$$

We derive the consumption plan by first observing that

$$W_k = \frac{W_{k+1}}{1+r} - e_k + c_k^*, \quad k \le T$$

Using this result repeatedly yields

$$W_k = \frac{W_{T+1}}{(1+r)^{T+1-k}} - \sum_{t=0}^{T-k} \frac{1}{(1+r)^t} e_{k+t} + \sum_{t=0}^{T-k} \frac{1}{(1+r)^t} c_{k+t}^*.$$

Using $k = 0$ and (**??**) gives $W_{T+1} = 0$. Taking conditional expectation,

$$W_k = -\sum_{t=0}^{T-k} \frac{1}{(1+r)^t} E[e_{k+t}|\mathcal{F}_k] + \sum_{t=0}^{T-k} \frac{1}{(1+r)^t} E[c_{k+t}^*|\mathcal{F}_k].$$

Using the earlier result, we obtain

$$\begin{aligned}W_k &= -\sum_{t=0}^{T-k} \frac{1}{(1+r)^t} E[e_{k+t}|\mathcal{F}_k] + \sum_{t=0}^{T-k} \frac{1}{(1+r)^t} c_k^* \\ &= -\sum_{t=0}^{T-k} \frac{1}{(1+r)^t} E[e_{k+t}|\mathcal{F}_k] + c_k^* \frac{1-(1+r)^{-(T+1-k)}}{1-(1+r)^{-1}},\end{aligned}$$

which, after solving for $c_k^*$, gives

$$c_k^* = \frac{r}{1+r-(1+r)^{-(T-k)}} \left( W_k + \sum_{t=0}^{T-k} \frac{1}{(1+r)^t} E[e_{k+t}|\mathcal{F}_k] \right).$$

*Note:* The solution is independent of the parameters of the parabola describing $U$.
*Remark:* This is a model for the *permanent income hypothesis* in macroeconomics, a theory which states that an individual's consumption is not only based on the current income, but instead the expected future income—the *permanent income*.