Corinne Emmenegger

# Introduction to Mathematical Finance: Sheet 13

## Exercise 2

**Prove that a monetary risk measure is quasi-convex iff it is convex.**

Definition 1: A functional $\rho : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is called $\boxed{\text{monetary risk measure}}$ if

1.) monotone (decreasing):

$$X \geq Y \Rightarrow \rho(X) \leq \rho(Y)$$

2.) cash invariant:

$$\forall m \in \mathbb{R} \ \forall X \in \mathcal{L}^\infty(\Omega, \mathcal{F}) : \rho(X+m) = \rho(X) - m$$

Definition 2: $\rho$ is $\boxed{\text{convex}}$ if $\forall X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}), \forall \lambda \in [0,1]$:

$$\rho(\lambda X + (1-\lambda) Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$$

Definition 3: $\rho$ is $\boxed{\text{quasi-convex}}$ if $\forall X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}) \ \forall \lambda \in [0,1]$:

$$\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}.$$

"$\Longleftarrow$"

- Let $\rho$ be a convex monetary risk measure.
  Let $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, $\lambda \in [0,1]$.

$$\Rightarrow \rho(\lambda X + (1-\lambda) Y) \underset{\rho \text{ convex}}{\leq} \lambda \rho(X) + (1-\lambda)\rho(Y) \underset{\lambda, (1-\lambda) \geq 0}{\leq} \lambda \max\{\rho(X), \rho(Y)\} + (1-\lambda) \cdot \max\{\rho(X), \rho(Y)\}$$

$$= \max\{\rho(X), \rho(Y)\}$$

$\hookrightarrow$ hence $\rho$ is quasi-convex.

$\boxed{\text{Question:}}$ To ensure $\tilde{X}, \tilde{Y} \in \mathcal{L}^\infty$, I needed that $\rho(X), \rho(Y) < \infty$. Why does this hold?

"$\Longrightarrow$"

- Let $\rho$ be a quasi-convex monetary risk measure.
- Let $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$

$$\Rightarrow \text{Define } \tilde{X} := X + \rho(X) \quad \text{and} \quad \tilde{Y} := Y + \rho(Y)$$

$$\Rightarrow \tilde{X}, \tilde{Y} \in \mathcal{L}^\infty(\Omega, \mathcal{F})$$

- Assume wlog that $\rho(\tilde{Y}) \geq \rho(\tilde{X})$. Otherwise rename everything.
- $\rho$ is quasi-convex, hence for $\lambda \in [0,1]$:

$$\Rightarrow \rho(\lambda \tilde{X} + (1-\lambda)\tilde{Y}) \leq \max\{\rho(\tilde{X}), \rho(\tilde{Y})\} = \rho(\tilde{Y})$$

$$\overset{\text{plug in}}{\Longleftrightarrow} \rho\Big(\lambda(X + \rho(X)) + (1-\lambda)(Y + \rho(Y))\Big) \leq \rho\big(Y + \rho(Y)\big)$$

$$\overset{\text{cash inv.}}{\Longleftrightarrow} \rho(\lambda X + (1-\lambda)Y) - \lambda\rho(X) - (1-\lambda)\rho(Y) \leq \rho(Y) - \rho(Y) = 0$$

$$\Longleftrightarrow \rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y) \rightsquigarrow \rho \text{ is convex.}$$

Exercise 3

Show that if $\rho$ is convex and normalized, then $\begin{cases} 1.)\ \rho(\lambda X) \leq \lambda\rho(X) & ,\ 0\leq\lambda\leq 1 \\ 2.)\ \rho(\lambda X) \geq \lambda\rho(X) & ,\ \lambda\geq 1 \end{cases}$

Definition 4: $\rho$ is $\boxed{\text{normalized}}$ if $\rho(0)=0$.

$\boxed{1}$

- Let $0\leq\lambda\leq 1$.
- $\rho$ is convex

$\Rightarrow \forall X,Y\in\mathcal{L}^\infty(\Omega,\mathcal{F}):\quad \rho(\lambda X+(1-\lambda)Y)\leq\lambda\rho(X)+(1-\lambda)\rho(Y)$

$\Rightarrow$ choosing $Y=0$ results in

$$\rho(\lambda X)=\rho(\lambda X+(1-\lambda)\cdot 0)\leq\lambda\rho(X)+(1-\lambda)\underbrace{\rho(0)}_{=0}=\lambda\rho(X).$$

$\boxed{2}$

- Let $\lambda\geq 1$
- Let $X\in\mathcal{L}^\infty(\Omega,\mathcal{F})$. I have

$$\rho(X)\underset{\lambda\neq 0}{=}\rho(\underbrace{\tfrac{1}{\lambda}}_{\in[0,1]}\lambda X)\overset{1.)}{\leq}\frac{1}{\lambda}\rho(\lambda X)$$

$$\overset{\lambda>0}{\Leftrightarrow}\lambda\rho(X)\leq\rho(\lambda X).$$

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## Exercise 4

convex & positive homogeneous

$\forall \lambda \geq 0 : \rho(\lambda X) = \lambda \rho(X)$.

Let $\rho$ be a coherent risk measure on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ and assume that $\rho$ admits a representation $\rho(X) = \sup_{Q \in \mathcal{Q}} E_Q[-X]$ for some class $\mathcal{Q} \subset \mathcal{M}_{1,f}$.

a) Show that $\rho(X) + \rho(-X) = 0$ for all $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

Let $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

$$\rho(X) + \rho(-X) = \sup_{Q \in \mathcal{Q}} E_Q[-X] + \sup_{Q \in \mathcal{Q}} E[X] \geq \sup_{Q \in \mathcal{Q}} \left\{ E_Q[-X] + E_Q[X] \right\}$$

$$= \sup_{Q \in \mathcal{Q}} \left\{ \underbrace{E_Q[-X+X]}_{=0} \right\} = 0$$

and for $X = 0$,

$$\rho(0) = \rho(0 \cdot 0) \underset{\substack{\uparrow \\ \rho \text{ pos.hom}}}{=} 0 \cdot \rho(0) = 0$$

$$\Rightarrow \rho(0) + \rho(-0) = \rho(0) + \rho(0) = 0 + 0 = 0.$$

$\Rightarrow$ therefore, $\rho(X) + \rho(-X) \geq 0$ and $\rho(0) + \rho(-0) = 0$ and thus $\rho(X) + \rho(-X) = 0$.

b) Show that the following conditions are equivalent:

1.) $\rho$ is additive, i.e. $\forall X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}): \rho(X+Y) = \rho(X) + \rho(Y)$

2.) $\forall X \in \mathcal{L}^\infty(\Omega, \mathcal{F}): \rho(X) + \rho(-X) = 0$

3.) The class $\mathcal{Q}$ reduces to a single element $Q$, i.e. $\rho$ is expected loss w.r.t. $Q$.

$\boxed{3.) \Rightarrow 2.)}$

- Let $\mathcal{Q} = \{Q\}$, and $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

$$\Rightarrow \rho(X) = E_Q[-X]$$

$$\Rightarrow \rho(X) + \rho(-X) = E_Q[-X] + E_Q[X] = E_Q[-X+X] = E_Q[0] = 0$$

$\boxed{2.) \Rightarrow 3.)}$

- Let $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $\rho(X) + \rho(-X) = 0$

$$\Rightarrow \rho(X) + \rho(-X) = \sup_{Q \in \mathcal{Q}} E_Q[-X] + \sup_{Q \in \mathcal{Q}} E_Q[X] \overset{!}{=} 0$$

$$\Leftrightarrow \sup_{Q \in \mathcal{Q}} E_Q[X] \overset{!}{=} - \sup_{Q \in \mathcal{Q}} E_Q[-X] = \inf_{Q \in \mathcal{Q}} \left( -E_Q[-X] \right) = \inf_{Q \in \mathcal{Q}} E_Q[X].$$

Question:

Why is $\mathcal{Q} \neq \emptyset$?

$\Rightarrow$ If the set $\mathcal{Q}$ had more than one element, sup and inf would not agree.

Therefore, $|\mathcal{Q}| \leq 1$.

$\boxed{3.) \Rightarrow 1.)}$

- Let $|\mathcal{Q}| = 1$ and $Q \in \mathcal{Q}$. Let $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

$$\Rightarrow \rho(X+Y) = E_Q[-X-Y] = E_Q[-X] + E_Q[-Y]$$
$$= \rho(X) + \rho(Y).$$

$\boxed{1.) \Rightarrow 2.)}$

- Let $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, and let $\rho$ be additive

$$\Rightarrow \rho(X) + \rho(-X) \underset{\rho \text{ additive}}{=} \rho(X-X) = \rho(0)$$

- Since $\rho$ is positive homogeneous,

$$\rho(0) = \rho(0 \cdot 0) = 0 \cdot \rho(0) = 0.$$

$\Rightarrow$ Hence,
$$\rho(X) + \rho(-X) = \rho(0) \overset{\checkmark}{=} 0.$$

$\Rightarrow$ I have shown

$$\begin{array}{l} 3.) \iff 2.) \\ \Downarrow \\ 1.) \Rightarrow 2.) \end{array}$$