ETH Zürich
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Prof. Josef Teichmann

Coordinator
Calypso Herrera

Introduction to mathematical finance

Exercise sheet 13

Exercise 13.1 Let $U : \mathbb{R} \to \mathbb{R}$ be a strictly increasing utility function and consider a general arbitrage-free market in T periods, with $\mathcal{F}_0$ trivial. Recall that $\mathcal{C} = \{(\xi \bullet X)_T\} - L^0_+$.

1. Show that an optimizer to

$$u(x) = \sup_{\xi \in \Xi} E\left[U(x + (\xi \bullet X)_T\right]$$

can be obtained from an optimizer of

$$u_{\mathcal{C}}(x) = \sup_{f \in \mathcal{C}} E\left[U(x + f)\right],$$

and vice versa.

2. Denote by $\mathbb{P}_a$ the set of absolutely continuous martingale measures. Show that if Ω is finite and $f \in L^0$, then

$$f \in \mathcal{C} \quad \Longleftrightarrow \quad E_Q[f] \leq 0, \quad \forall Q \in \mathbb{P}_a.$$

Exercise 13.2 Prove that a monetary risk measure is quasi-convex if and only if it is convex.

Exercise 13.3 Show that if ρ is convex and normalized, then

$$\begin{aligned} \rho(\lambda X) &\leq \lambda\rho(X) \quad \text{for} \quad 0 \leq \lambda \leq 1, \\ \rho(\lambda X) &\geq \lambda\rho(X) \quad \text{for} \quad \lambda \geq 1. \end{aligned}$$

Exercise 13.4 Let ρ be a coherent risk measure on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ and assume that ρ admits a representation

$$\rho(X) = \sup_{Q \in \mathcal{Q}} E_Q(-X)$$

with some class $\mathcal{Q} \subset \mathcal{M}_{1,f}$.

(a) Show that $\rho(X) + \rho(-X) \geq 0$ for all $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

(b) Show that the following condition are equivalent.

1. ρ is *additive*, i.e., $\rho(X + Y) = \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.
2. $\rho(X) + \rho(-X) = 0$ for all $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$
3. The class Q reduces to a single element Q, i.e., ρ is simply expected loss with respect to Q.