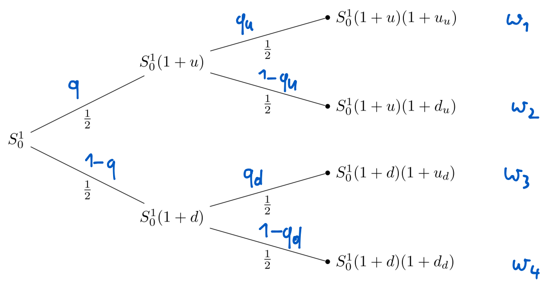
Exercise 11.2

Consider the asset given by the following tree:



where the fractions denote probabilities. Let $S_k^0 = (1+r)^k$ and

$$-1 < r \in (d,u) \cap (d_u,u_u) \cap (d_d,u_d) \neq \emptyset,$$

i.e., the market is arbitrage-free.

Strategies here can be identified with vectors in $\mathbb{R}^3$ via $\xi = (\xi_1, \xi_2^u, \xi_2^d)$. Find the optimizer ξ^* to the problem of maximizing (exponential) utility of final wealth:

$$\max_{\xi \in \mathbb{R}^3} E\left[1 - \exp\left(-v_0 - (\xi \bullet X)_2\right)\right].$$

<u>sol</u>: $\rightsquigarrow$ $U(x) = 1 - \exp(-x)$ is the Utility function

$\Rightarrow$ the conjugate function is $V(y) = \sup_{x\in\mathbb{R}} \left(U(x) - xy\right) = \sup_{x\in\mathbb{R}} \left(1-\exp(-x) - xy\right) = (\mathrm{I})$

$\rightsquigarrow$ $\frac{d}{dx}\left(1-\exp(-x)-xy\right) = \exp(-x) - y \overset{!}{=} 0 \iff x = -\ln(y)$ is maximizer since $U(x) - xy$ is strictly concave

$\rightarrow$ $V(y) = (\mathrm{I}) = 1 - \exp(\ln(y)) + \ln(y)y = 1 + y(\ln(y) - 1)$

$\rightsquigarrow$ using the dual theory we have: $u(v_0) = \sup_{\xi\in\mathbb{R}^3} \mathbb{E}_P\left(U(v_0 + (\xi\bullet X)_2)\right) =$

$= \sup_{Y\in L^0,\ \mathbb{E}_Q(Y)\leq v_0\ \forall Q\in\mathcal{M}^a} \mathbb{E}_P\left(U(Y)\right) \overset{!}{=} \inf_{z>0}\Big(\underbrace{\inf_{Q\in\mathcal{M}^a}\left(\mathbb{E}_P\left(V\left(z\frac{dQ}{dP}\right)\right)\right)}_{=v(z)} + z v_0\Big) =$

$= \inf_{z>0}\left(v(z) + z v_0\right)$ (3)

$\rightsquigarrow$ first compute $\mathcal{M}^a$ $\rightsquigarrow$ every $Q\in\mathcal{M}^a$ can be described as (q, q_u, q_d) as in the sketch $\rightsquigarrow$ from the mg.-property of $X_t^1 = \left(\frac{S_t^1}{S_t^0}\right) = \frac{S_t^1}{(1+r)^t}$ we get the following equations:

(1): $S_0^1 = X_0^1 \overset{\text{mg. prop.}}{=} \mathbb{E}_Q(X_1^1 \mid \mathcal{F}_0) = \frac{1}{(1+r)} \cdot S_0^1 \cdot \mathbb{E}_Q(1+R_1^1 \mid \mathcal{F}_0) = \mathbb{E}_Q(1+R_1^1) = q(1+u) + (1-q)(1+d)$

$\iff 1+r = q(1+u) + (1-q)(1+d) \iff r = qu + (1-q)d \iff q = \frac{r-d}{u-d}$

(2): $S_1^1 \frac{1}{(1+r)} = X_1^1 \overset{\text{mg. prop}}{=} \mathbb{E}_Q(X_2^1 \mid \mathcal{F}_1) = \frac{1}{(1+r)^2} S_1^1\, \mathbb{E}_Q\left((1+R_2^1) \mid \mathcal{F}_1\right) \iff (1+r) = \mathbb{E}_Q(1+R_2^1 \mid \mathcal{F}_1)$

$\rightsquigarrow$ conditioning on what S_1^1 is, we get the following 2 equations:

(2a): $(1+r) = q_u(1+u_u) + (1-q_u)(1+d_u) \iff q_u = \frac{r-d_u}{u_u-d_u}$, (here we evaluated (2) on $\{S_1^1 = S_0^1(1+u)\} \subset \mathcal{F}_1$)

(2b): $(1+r) = q_d(1+u_d) + (1-q_d)(1+d_d) \iff q_d = \frac{r-d_d}{u_d-d_d}$, (here we evaluated (2) on $\{S_1^1 = S_0^1(1+d)\} \subset \mathcal{F}_1$)

$\Rightarrow$ $\mathcal{M}^a = \mathcal{M}^e = \left\{(q, q_u, q_d) \mid q_i = \frac{r-d_i}{u_i-d_i}\right\}$ $\Rightarrow$ the market is complete

$\Rightarrow$ (3) simplifies to: $u(v_0) = \inf_{z>0}\Big(\underbrace{\mathbb{E}_P\left(V\left(z\frac{dQ}{dP}\right)\right)}_{=v(z)} + z v_0\Big)$ $\rightsquigarrow$ minimizer is given by $\hat{z}$ with $v'(\hat{z}) = -v_0$, since $v(\cdot)$ is convex

$\rightsquigarrow$ $v(z) = \sum_{i=1}^4 p_i V\left(z\frac{q_i}{p_i}\right)$ $\rightsquigarrow$ $v'(z) = \sum_{i=1}^4 p_i V'\left(z\frac{q_i}{p_i}\right)\cdot\frac{q_i}{p_i} = \sum_{i=1}^4 q_i V'\left(z\frac{q_i}{p_i}\right) = \sum_{i=1}^4 q_i \ln\left(z\frac{q_i}{p_i}\right) = \sum_{i=1}^4 q_i\left(\ln(z) + \ln\left(\frac{q_i}{p_i}\right)\right) =$

$= \ln(z) + \underbrace{\sum_{i=1}^4 q_i \ln\left(\frac{q_i}{p_i}\right)}_{=c_0} =: \ln(z) + c_0$

$V'(z) = \frac{d}{dz}\left(1 + z(\ln(z) - 1)\right) = (\ln(z) - 1) + z\frac{1}{z} = \ln(z)$

$\rightsquigarrow$ here we use: $q_1 = q q_u$, $q_2 = q(1-q_u)$, $q_3 = (1-q)q_d$, $q_4 = (1-q)(1-q_d)$

$\rightsquigarrow$ $v'(\hat{z}) = -v_0 \iff \ln(\hat{z}) + c_0 = -v_0 \iff \hat{z} = \exp(-v_0 - c_0)$

$\Rightarrow$ $u(v_0) = v(\hat{z}) + \hat{z}v_0 = \hat{z}v_0 + \mathbb{E}_P\left(V\left(\hat{z}\frac{dQ}{dP}\right)\right) = \hat{z}v_0 + \sum_{i=1}^4 p_i \sup_{Y^i\in\mathbb{R}}\left(U(Y^i) - \hat{z}\frac{q_i}{p_i}Y^i\right)$ $\quad (U')^{-1} = -V'$

$\rightsquigarrow$ $\hat{Y}^i$ is therefore given by $U'(\hat{Y}^i) = \hat{z}\frac{q_i}{p_i} \iff \hat{Y}^i = (U')^{-1}\left(\hat{z}\frac{q_i}{p_i}\right) = -V'\left(\hat{z}\frac{q_i}{p_i}\right) = -\ln\left(\hat{z}\frac{q_i}{p_i}\right) = -\ln(\hat{z}) - \ln\left(\frac{q_i}{p_i}\right) =$

$= v_0 + c_0 - \ln\left(\frac{q_i}{p_i}\right)$

$\rightsquigarrow$ $\hat{Y}$ given by $\hat{Y}(\omega_i) = \hat{Y}^i$ is the optimal final wealth we can reach starting from v_0 (optimal in the sense that it optimizes the expected utility

$\rightsquigarrow$ to find $\hat{\xi}$ we take a RV. $\hat{Z}$ with $\hat{Z}_2 = \hat{Y}$ and $\hat{Z}_t = \overbrace{v_0 + (\hat{\xi}\bullet X)_t}^{=\hat{Z}_{t-1} + \hat{\xi}_t(X_t^1 - X_{t-1}^1)}$ and compute $\hat{\xi}$ using that $(\hat{\xi}\bullet X)$ is a mg hence also $\hat{Z}$

$\rightsquigarrow$ $\hat{Z}_2 = \hat{Y}$ is known, $\hat{Z}_1(\{\omega_1,\omega_2\}) = \mathbb{E}(\hat{Z}_2 \mid \mathcal{F}_1)(\{\omega_1,\omega_2\}) = q_u\hat{Y}^1 + (1-q_u)\cdot\hat{Y}^2$, analogously: $\hat{Z}_1(\{\omega_3,\omega_4\}) = q_d\hat{Y}^3 + (1-q_d)\hat{Y}^4$

and $\hat{Z}_0 = q\hat{Z}^1 + (1-q)\hat{Z}^2$ $\rightsquigarrow$ these are the discounted prices and $\hat{Z}_0 = v_0$ has to hold since v_0 is the starting capital

$\rightsquigarrow$ we can write: $\hat{Z}_1(\omega_1) + \hat{\xi}_u(X_2^1 - X_1^1)(\omega_1) = \hat{Z}_2(\omega_1) = \hat{Y}^1 \iff \hat{\xi}_u = \frac{\hat{Y}^1 - \hat{Z}_1(\omega_1)}{(X_2^1 - X_1^1)(\omega_1)}$

$\rightsquigarrow$ analogously we find $\hat{\xi}_d = \frac{\hat{Y}^3 - \hat{Z}_1(\omega_3)}{(X_2^1 - X_1^1)(\omega_3)}$, $\hat{\xi}_1 = \frac{\hat{Z}_1(\omega_1) - \overbrace{\hat{Z}_0(\omega_1)}^{=v_0}}{(X_1^1 - X_0^1)(\omega_1)}$ ∎

$\hat{\xi}_u = \frac{1}{(X_2^1 - X_1^1)(\omega_1)}(1-q_u)(\hat{Y}_1 - \hat{Y}_2) = \frac{(1+r)^2}{S_0^1(1+u)\cancel{(u_u-r)}}\Big(\overbrace{\frac{u_u-d_u}{u_u-d_u} - \frac{r-d_u}{u_u-d_u}}^{=\frac{u_u-r}{u_u-d_u}}\Big)\left(\ln\left(\frac{q_2}{p_2}\right) - \ln\left(\frac{q_1}{p_1}\right)\right) = \frac{(1+r)^2}{S_0^1(1+u)(u_u-d_u)}\cdot\overbrace{\ln\left(\frac{q_2}{q_1}\cdot\frac{\cancel{p_1}}{\cancel{p_2}}\right)}^{=\ln\left(\frac{1-q_u}{q_u}\right)} = \frac{(1+r)^2}{S_0^1(1+u)(u_u-d_u)}\cdot\ln\left(\frac{u_u-r}{u_u-d}\cdot\frac{u_u-d_u}{r-d_u}\right)$

$= \frac{(1+r)^2}{S_0^1(1+u)(u_u-d_u)}\ln\left(\frac{u_u-r}{r-d_u}\right)$

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Exercise 12.1 Consider the arbitrage-free market in T periods with a riskless bond with zero interest rate. Assume that $H \in L^0_+(\mathcal{F}_T)$ is an attainable claim and fix one EMM Q. Let U be an exponential utility function, $\mathcal{A} = \Theta$ be the set of predictable processes, and consider the two functions

$$u(x) = \max_{\xi \in \mathcal{A}} E[U(x + (\xi \bullet X)_T)]$$

and

$$u_H(x) = \max_{\xi \in \mathcal{A}} E[U(x + (\xi \bullet X)_T - H)].$$

Given a wealth level x, the *utility indifference price* $p_H(x)$ of H is defined as the solution to

$$u(x) = u_H(x + p_H(x)). \quad (*)$$

1. Show that $E_Q[H]$ is the unique solution to the above equation.
2. Can the assumptions on the utility function be generalized?

a.) Proof: $\leadsto$ w.l.o.g. $U(x) = -e^{-\alpha x}$, $\alpha > 0$ (since adding a const doesn't change the maximization problem)

$\leadsto$ H attainable $\Rightarrow \exists\, \xi_H \in \mathcal{A}: \; H = \mathbb{E}_Q(H) + (\xi_H \bullet X)_T \quad (1)$

$$\Rightarrow u_H(x) \overset{(1)}{=} \max_{\xi \in \mathcal{A}} \mathbb{E}\left[U\left((x - \mathbb{E}_Q(H)) + ((\xi - \xi_H)\bullet X)_T\right)\right] = \quad \xi - \xi_H = \eta \in \mathcal{A} \;\; \forall \xi \in \mathcal{A}$$

$$= \max_{\eta \in \mathcal{A}} \mathbb{E}\left[U\left((x - \mathbb{E}_Q(H)) + (\eta \bullet X)_T\right)\right] = u(x - \mathbb{E}_Q(H)) \quad (2)$$

$\leadsto$ from (2) we immediately get: $u_H(x + \mathbb{E}_Q(H)) = u(x) \Rightarrow \mathbb{E}_Q(H)$ is a solution for: [find $p_H(x)$ s.t.: $u(x) = u_H(x + p_H(x))$]

Claim 1: $\mathbb{E}_Q(H)$ is the unique solution. Proof 1: let $y > 0$ and set $z := y + \mathbb{E}_Q(H) \Rightarrow u_H(x+z) \overset{(2)}{=} u(x + z - \mathbb{E}_Q(H)) = u(x+y) =$

$$= \max_{\xi \in \mathcal{A}} \mathbb{E}\left[U(x + y + (\xi \bullet X)_T)\right] \underset{U(x) = -\exp(-\alpha x)}{=} \max_{\xi \in \mathcal{A}} \mathbb{E}\Big(\underbrace{U(x + (\xi\bullet X)_T)}_{<0,\text{ since } U<0} \cdot \underbrace{\exp(-\alpha y)}_{\in(0,1),\text{ since } y>0}\Big) \underset{\text{use that a maximizer } \hat\xi \text{ exists} \Rightarrow \max \mathbb{E}(U(\dots)) < 0}{>} \max_{\xi\in\mathcal{A}} \mathbb{E}\left(U(x + (\xi \bullet X)_T)\right) = u(x)$$

$\leadsto$ in the same way we have for $y < 0$ and $z = y + \mathbb{E}_Q(H)$ that $u_H(x+z) < u(x)$ $\quad \square$ (Claim 1) $\quad \blacksquare$

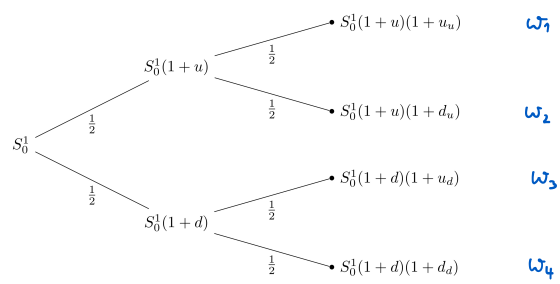
b) sol: $\leadsto$ we only need the property that $u(\cdot)$ is strictly increasing to get that $\mathbb{E}_Q(H)$ is the unique solution

$\leadsto$ so we can take any utility function U satisfying that its value function u is strictly increasing on $\mathrm{dom}(U)$

$\leadsto$ obviously, if $x \notin \mathrm{dom}(U)$ (in the case that $\mathrm{dom}(U) \neq \mathbb{R}$) then the solution to $(*)$ is not unique since $u(x) = -\infty$ in this case

Exercise 12.2

Recall the market structure from Exercise 11.2:



where the fractions denote probabilities. Let $S_k^0 = (1+r)^k$ and

$$-1 < r \in (d, u) \cap (d_u, u_u) \cap (d_d, u_d) \neq \emptyset.$$

Strategies are identified with vectors in $\mathbb{R}^3$ via $\xi = (\xi_1, \xi_2^u, \xi_2^d)$. Find the optimizer ξ^* to the problem

$$\max_{\xi \in \mathbb{R}^3} E\left[1 - \exp\left(-v_0 - (\xi \bullet X)_2\right)\right].$$

using dynamic programming[1].

sol: $\leadsto$ we use backward recursion, meaning that we first try to find optimal investment strategies, when we already are at time $t = T-1$ and then go back to earlier times recursively $\leadsto$ the resulting strategy will then be optimal in total since at every time step we look for the optimal way to invest now, knowing that in the following time steps we are investing optimally

$\leadsto$ if we are on the set $\{\omega_1, \omega_2\}$, then for given starting capital x_1 (at time $t=1$) we want to maximize:

$$\mathbb{E}\left[1 - \exp\left(-(x_1 + \xi_u \cdot (X_2 - X_1))\right) \mid S_1^1 = S_0^1(1+u)\right] = (\mathrm{I}) \text{ over } \xi_u \in \mathbb{R}$$

$\leadsto$ $(\mathrm{I}) = 1 - \overbrace{\exp(-x_1)}^{>0} \cdot \mathbb{E}\left[\exp(-\xi_u(X_2 - X_1)) \mid S_1^1 = S_0^1(1+u)\right]$

$\Rightarrow$ maximizing (I) is equivalent to minimizing:

$$\mathbb{E}\left(\exp(-\xi_u(X_2 - X_1)) \mid S_1^1 = S_0^1(1+u)\right) = \frac{1}{2}\exp\left(-\xi_u\left(\frac{S_0^1(1+u)(1+u_u)}{(1+r)^2} - \frac{S_0^1(1+u)}{(1+r)}\right)\right) +$$

$$+ \frac{1}{2}\exp\left(-\xi_u \frac{S_0^1(1+u)}{(1+r)}\left(\frac{1+d_u}{1+r} - 1\right)\right) =: \frac{1}{2} g_1(\xi_u)$$

$$\leadsto g_1'(\hat\xi_u) = -\frac{S_0^1(1+u)}{(1+r)^2}\left((u_u - r)\exp\Big(-\hat\xi_u \overbrace{\frac{S_0^1(1+u)}{(1+r)^2}}^{=:A}(u_u - r)\Big) + (d_u - r)\exp\left(-\hat\xi_u \frac{S_0^1(1+u)}{(1+r)^2}(d_u - r)\right)\right) \overset{!}{=} 0$$

$$\Leftrightarrow (u_u - r)\exp(-\hat\xi_u A(u_u - r)) = -(d_u - r)\exp(-\hat\xi_u A(d_u - r)) \Leftrightarrow -\frac{u_u - r}{d_u - r} = \exp\Big(\hat\xi_u A \overbrace{(u_u - r - d_u + r)}^{=u_u - d_u}\Big) \Leftrightarrow \hat\xi_u = \ln\left(\frac{u_u - r}{r - d_u}\right) \cdot \frac{(1+r)^2}{S_0^1(1+u)(u_u - d_u)}$$

using that $d_u < r < u_u$

$\leadsto$ we have that $g_1(\xi) \xrightarrow{\xi \to \pm\infty} \infty \Rightarrow \hat\xi_u$ has to be a minimizer (and therefore the unique minimizer) of g_1

$\leadsto$ similarily we get $\hat\xi_d = \ln\left(\frac{u_d - r}{r - d_d}\right) \cdot \frac{(1+r)^2}{S_0^1(1+d)(u_d - d_d)}$ as optimal strategy when starting from the set $\{\omega_3, \omega_4\}$

$\leadsto$ we see that $\hat\xi_u, \hat\xi_d$ are independent of the starting capital x_1

$\leadsto$ $\hat\xi_1$ similarily has to minimize:

$$\mathbb{E}\left(\exp(-\xi_1(X_1 - X_0) - \hat\xi_2(X_2 - X_1))\right) = \frac{1}{4}\exp\left(-\xi_1 S_0^1\left(\frac{1+u}{1+r} - 1\right)\right) \cdot \overbrace{\left[\exp\left(-\hat\xi_u \frac{S_0^1(1+u)}{(1+r)^2}(u_u - r)\right) + \exp\left(-\hat\xi_u \frac{S_0^1(1+u)}{(1+r)^2}(d_u - r)\right)\right]}^{=:B}$$

$$+ \frac{1}{4}\exp\left(-\xi_1 S_0^1\left(\frac{1+d}{1+r} - 1\right)\right) \cdot \underbrace{\left[\exp\left(-\hat\xi_d \frac{S_0^1(1+d)}{(1+r)^2}(u_d - r)\right) + \exp\left(-\hat\xi_d \frac{S_0^1(1+d)}{(1+r)^2}(d_d - r)\right)\right]}_{=:C} = \frac{1}{4}\exp\left(-\xi_1 S_0^1\left(\frac{u-r}{1+r}\right)\right) \cdot B + \frac{1}{4}\exp\left(-\xi_1 S_0^1\left(\frac{d-r}{1+r}\right)\right) \cdot C =: \frac{1}{4} g_3(\xi_1)$$

$$\leadsto g_3'(\hat\xi_1) = -S_0^1\left(\frac{u-r}{1+r}\right) B \exp\left(-\hat\xi_1 S_0^1\left(\frac{u-r}{1+r}\right)\right) - S_0^1\left(\frac{d-r}{1+r}\right) \cdot C \cdot \exp\left(-\hat\xi_1 \cdot S_0^1\left(\frac{d-r}{1+r}\right)\right) \overset{!}{=} 0$$

$$\Leftrightarrow -(u-r) B \exp\left(-\hat\xi_1 S_0^1 \frac{u-r}{1+r}\right) = (d-r) C \exp\left(-\hat\xi_1 S_0^1 \frac{d-r}{1+r}\right) \Leftrightarrow \frac{u-r}{r-d} \cdot \frac{B}{C} = \exp\left(\hat\xi_1 \frac{S_0^1}{1+r} \cdot (u-d)\right) \Leftrightarrow \hat\xi_1 = \ln\left(\frac{u-r}{r-d} \cdot \frac{B}{C}\right) \cdot \frac{1+r}{S_0^1(u-d)}$$

$\leadsto$ again we have that $g_3(\xi) \xrightarrow{\xi \to \pm\infty} \infty$, since $d < r < u$ $\Rightarrow \hat\xi_1$ is minimizer of g_3 and unique

$\rightarrow (\hat\xi_1, \hat\xi_u, \hat\xi_d) = \hat\xi$ is the unique maximizer respectively the optimal investment strategy $\quad \blacksquare$