Moritz Weiss

IMF Sheet 13

bounded functions

13.2 "$\Leftarrow$": Suppose the risk measure is convex. Choose $X, Y \in L^\infty$, $\lambda \in [0,1]$ then:

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda \underbrace{\rho(X)}_{\leq \rho(X) \vee \rho(Y)} + (1-\lambda)\underbrace{\rho(Y)}_{\leq \rho(X)\vee\rho(Y)} \leq \rho(X) \vee \rho(Y).$$

"$\Rightarrow$": Suppose the risk measure is quasi-convex: Choose $X, Y \in L^\infty$, $\lambda \in [0,1]$.

We need to show: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.

By cash additivity this is equivalent to:

$$\rho(\lambda(X+\rho(X)) + (1-\lambda)(Y+\rho(Y))) \leq 0 \quad (*)$$

By quasi convexity we have:

$$\rho(\lambda(X+\rho(X)) + (1-\lambda)(Y+\rho(Y))) \leq \max(\lambda \underbrace{\rho(X+\rho(X))}_{=0}, (1-\lambda)\underbrace{\rho(Y+\rho(Y))}_{=0}) = 0$$

Hence $(*)$ is true.

13.3 $$\rho(\lambda X + (1-\lambda)\cdot 0) \underset{\text{convexity}}{\leq} \lambda\rho(X) + (1-\lambda)\rho(0) \overset{\rho \text{ normalized}}{=} \lambda\rho(X).$$

$\lambda \in [0,1]$

Choose $\lambda \in [1,\infty)$, then: $0 < \frac{1}{\lambda} \leq 1$.

By the first part:

$$\rho\left(\frac{1}{\lambda}\lambda X\right) \leq \frac{1}{\lambda}\rho(\lambda X)$$

$$\Rightarrow \lambda\rho(X) \leq \rho(\lambda X).$$

13.4 (a) $$\rho(X) + \rho(-X) \geq \sup_{Q\in\mathcal{Q}}\left(E_Q[-X] + E_Q[X]\right) = 0$$

(b) 1. $\Rightarrow$ 2. $\rho(X) + \rho(-X) = \rho(0) = 0$

2. $\Rightarrow$ 3. for $X \in L^\infty(\Omega, \mathcal{F})$ we have: $\longrightarrow$

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X] = -\sup_{Q \in \mathcal{Q}} \left(-\mathbb{E}_Q[-X]\right) = \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X]$$

This implies for any $\tilde{Q} \in \mathcal{Q}$:

$$\rho(X) = \sup_{Q \in Q} \mathbb{E}_Q[-X] = \mathbb{E}_{\tilde{Q}}[-X] .$$

3. $\Rightarrow$ 1. $\quad \rho(X+Y) = \mathbb{E}_{\tilde{Q}}[-(X-Y)] = \rho(X) + \rho(Y)$