Algebraic Curves (FS 2017)

Prof. R. Pandharipande
J. Schmitt

Self-study: Sheaves and Sheaf cohomology

The goal of this exercise sheet is to allow you to learn about sheaves and their cohomology groups. The main reference for this is Forster, Lectures on Riemann Surfaces. Below we will give section numbers in Forster which you should read and then some exercises to practice and to work towards understanding the final step of the proof of the Abel-Jacobi theorem. Sections indicated with a † are optional and may require additional reading on your part to understand the notation.

1. ((Pre)sheaves and stalks)

a) Read sections 6.1-6.6.

b) Given an abelian group G and a topological space X with a point $p \in X$, define a presheaf G_p on X by

$$G_p(U) = \begin{cases} G, p \in U, \\ 0, p \notin U \end{cases} \quad \text{and } \rho_V^U(a) = \begin{cases} a, p \in V \subset U, \\ 0, p \notin V \end{cases}$$

for $V \subset U \subset X$ open sets. Verify that this defines a sheaf on X. Compute its stalk at a point $q \in X$.

c) Given a smooth, connected curve C, define the sheaf Ω_C of holomorphic differential forms on C. For $P = \{p_1, \ldots, p_n\} \subset C$ a finite set, define the sheaf $\widehat{\Omega}_C$ of meromorphic differential forms on C with at most simple poles at the elements of P. Convince yourself that both of them are sheaves.

d) For two (pre)sheaves $\mathcal{F}, \mathcal{G}$ of abelian groups on X define their direct sum $\mathcal{F} \oplus \mathcal{G}$ by

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

and the obvious restriction maps. Convince yourself that $\mathcal{F} \oplus \mathcal{G}$ is again a (pre)sheaf. Similarly one defines direct sums $\bigoplus_{i=1}^n \mathcal{F}_i$ of sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_n$.

2. (Sheaf cohomology groups)

a) Read sections 12.1 - 12.5, 12.6†, 12.7-12.8 (statements, proofs = †), 12.10.

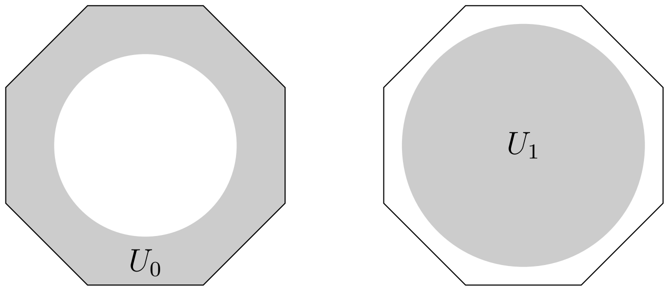
b) Let $\mathcal{F}$ be a sheaf on the one-point space $X = \{pt\}$. Show that $H^1(X, \mathcal{F}) = 0$.

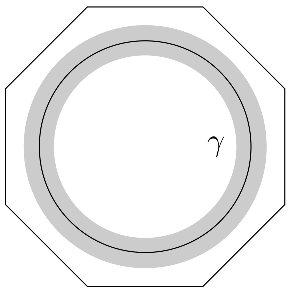
c) Let X be a topological space, $p \in X$, G an abelian group. Show that $H^1(X, G_p) = 0$.

d)* Recall Theorem 12.8 in Forster. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the circle and let $\mathbb{Z}$ be the sheaf of locally constant functions to $\mathbb{Z}$ on S^1. Show that $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$. You can assume that for the topological space $X = [0, 1]$ one has $H^1(X, \mathbb{Z}) = 0$.

3. (Computing $H^1(C, \Omega_C)$)

a) Recall that a curve C of genus $g > 0$ can be obtained by gluing the edges of a $4g$-gon in the plane in a suitable pattern. Consider the open cover $\mathcal{U} = \{U_0, U_1\}$ of C illustrated by the following picture in the case $g = 2$:





$U_0 \cap U_1$

Let γ be a closed path in $U_0 \cap U_1$ as indicated above. An element of $H^1(\mathcal{U}, \Omega_C)$ is represented by a differential form $f(z)dz$ on $U_0 \cap U_1$. Show that the map

$$H^1(\mathcal{U}, \Omega_C) \to \mathbb{C}, [f(z)dz] \mapsto \int_\gamma f(z)dz$$

is well-defined and surjective.

b) Using (without proof) the fact that $H^1(C, \Omega_C)$ is one-dimensional, show that the map from a) induces an isomorphism $H^1(C, \Omega_C) \xrightarrow{\sim} \mathbb{C}$. (*Hint:* Section 12.5)

4. (Morphisms and exact sequences of sheaves)

a) Read sections 15.1, 15.2†, 15.3-15.8, 15.9†.

b) In the situation of Exercise 1. c) consider the sequence

$$0 \to \Omega_C \xrightarrow{\varphi} \widehat{\Omega}_C \xrightarrow{\psi} \bigoplus_{i=1}^{n} \mathbb{C}_{p_i} \to 0,$$

where $\varphi(U)$ is the map identifying a holomorphic differential on $U \subset C$ with a meromorphic differential on U with at most simple poles at P and where $\psi(U)(\omega) = (\mathrm{Res}_{p_1}\omega, \ldots, \mathrm{Res}_{p_n}\omega)$. Here $\mathrm{Res}_{p_i}\omega = 0$ if $p_i \notin U$. Show that this sequence is exact.

5. (Long exact sequence in cohomology)

a) Read sections 15.10-15.11, 15.12 (statement, proof= †).

b) The short exact sequence of Exercise 4. b) induces a long exact sequence

$$0 \to H^0(C, \Omega_C) \to H^0(C, \widehat{\Omega}_C) \to H^0(C, \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}) \xrightarrow{\delta^*} H^1(C, \Omega_C) \to \cdots$$

We want to understand the map δ^*. Use the natural isomorphism

$$H^0(C, \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}) \cong \mathbb{C}^n$$

and the isomorphism $H^1(C, \Omega_C) \cong \mathbb{C}$ from Exercise 3. b) to show that δ^* is given by

$$\delta^*(r_1, \ldots, r_n) = 2\pi i (r_1 + \ldots + r_n).$$

Hint: When going through the construction of δ^* in Section 15.11 of Forster, you can use the cover $\mathcal{U}$ of C from Exercise 3. a) if you make sure that all points of P are contained in U_1 but none of them in U_0.

c) Use the full exact sequence from part b) to show

$$H^1(C, \widehat{\Omega}_C) = H^1(C, \bigoplus_{i=1}^{n} \mathbb{C}_{p_i}) = 0.$$

You can hand in the sheet until March 30 if you want feedback.

Exercises with * are possibly harder and should be considered as optional challenges.