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Exercise Sheet 1

1. Show that every regular (holomorphic) function on a nonsingular, connected projective curve C is constant. Can you write down two proofs, once seeing C as an algebraic variety and once as a complex manifold?

2. Consider the meromorphic function $f(z) = \frac{z^3 - z^2}{z^2+1}$ on $\mathbb{C}$.

i) Using the identification $\mathbb{C} \subset \mathbb{CP}^1$, $z \mapsto [1 : z]$, on source and target, identify f as a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ and write it in the form $[x : y] \mapsto [F(x, y), G(x, y)]$ for homogeneous polynomials F, G of the same degree, having no common zeroes on $\mathbb{CP}^1$.

ii) Compute $\mathrm{div}(f)$.

3. Let $\Lambda \subset \mathbb{C}$ be a lattice (i.e. $\Lambda = \mathbb{Z}v + \mathbb{Z}w$ for $v, w \in \mathbb{C}$ two $\mathbb{R}$-linearly independent vectors).

i) Convince yourself that $\mathbb{C}/\Lambda$ with the quotient topology is a torus (i.e. a compact oriented surface of genus 1) and that it has the structure of a complex manifold of dimension 1, i.e. an algebraic curve.

ii) A meromorphic function f on $\mathbb{C}$ is called *doubly periodic* (with respect to Λ) if $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$. Show that every holomorphic doubly periodic function $f : \mathbb{C} \to \mathbb{C}$ is constant.

iii) Assume f is doubly periodic with no poles on the boundary of the *fundamental domain*

$$D = \{\lambda v + \mu w : 0 \leq \lambda, \mu \leq 1\}.$$

Show that the sum of the residues of all poles of f inside D is zero. Conclude that no such f can have a unique simple pole in D.

iv) The Weierstrass $\wp$-function of the lattice Λ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left[\frac{1}{(z - w)^2} - \frac{1}{w^2} \right] \text{ for } z \in \mathbb{C} \setminus \Lambda.$$

With some work you can show that this series converges absolutely for every $z \in \mathbb{C} \setminus \Lambda$ and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$, so $\wp$ defines a meromorphic function with double poles at the elements of Λ.

Show that the derivative $\wp'$ is doubly periodic and odd ($\wp'(-z) = -\wp'(z)$) and that $\wp$ is even ($\wp(-z) = \wp(z)$). Use this to show that $\wp$ is doubly periodic.

v) Show that $\wp'$ has simple zeroes exactly at the points $\frac{v}{2} + \Lambda$, $\frac{w}{2} + \Lambda$, $\frac{v+w}{2} + \Lambda$.
Hint: Here you can use that $\wp' : \mathbb{C}/\Lambda \dashrightarrow \mathbb{C}$ satisfies $\deg(\mathrm{div}(\wp')) = 0$.

vi) Prove that every doubly periodic function f with double poles exactly at the elements of Λ is of the form $f(z) = a\wp(z) + b$ for $a \in \mathbb{C}^*, b \in \mathbb{C}$. So up to translation ($z \mapsto z + z_0$), scaling and adding constant functions, $\wp$ is the unique doubly periodic function having as poles the Λ-translates of one double pole!

vii) Let the Laurent expansion of (the even function) $\wp$ around $z = 0$ be given by

$$\wp(z) = \frac{1}{z^2} + c + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6).$$

Show that $c = 0$ and that $\wp$ satisfies the differential equation

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp(z) - g_3.$$

Hint: Use part ii).

Thus we have a holomorphic map

$$g : \mathbb{C}/\Lambda \to E = V(Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3) \subset \mathbb{CP}^2, z \mapsto [\wp(z) : \wp'(z) : 1],$$

with coordinates $[X : Y : Z]$ on $\mathbb{CP}^2$, sending $z = 0$ to $[0 : 1 : 0]$.

viii)* Show that E is a smooth algebraic curve if $(g_2)^3 - 27(g_3)^2 \neq 0$ (if you are unfamiliar with such calculations in projective space, just show that $V(y^2 - 4x^3 + g_2x + g_3) \subset \mathbb{C}^2$ is smooth in this case).

ix)* Conclude that g is an isomorphism.

Due March 2.

Exercises with * are possibly harder and should be considered as optional challenges.