Algebraic Curves (FS 2017)

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Exercise Sheet 2

1. Show that for curves C_g, C_h of genus g, h respectively there does not exist a non-constant holomorphic map $f : C_g \to C_h$ if $g < h$. *Hint:* Given the results from the lecture, the proof is very short, so don't try something complicated!

2. In the lecture you saw the following (seeming) contradiction to the Riemann-Hurwitz formula: given a function $f : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ with $f^{-1}([\infty]) = d[0]$, it looks like $f(z) = \frac{c}{z^d} + \ldots$ around $z = 0$, so $f'(z) = \frac{-cd}{z^{d+1}} + \ldots$. Computing the divisor we have

$$\mathrm{div}(f') = [p_1] + \ldots [p_b] - (d+1)[0].$$

As this should have degree 0 we know that (counted with multiplicities) f' has $b = d + 1$ zeroes, which are the ramification points of f. However, as the genus of $\mathbb{C}/\Lambda$ is 1 and the genus of $\mathbb{CP}^1$ is 0, Riemann-Hurwitz tells us that $2 - 2 = d(-2) + b$, so we expect a total of $b = 2d$ ramification points. What is wrong in this counter-example?

3. In this exercise we want to recall the notion of a holomorphic differential form. Let C be a complex manifold of dimension 1 and let $(\varphi_i : U_i \xrightarrow{\sim} W_i \subset C)_{i\in I}$ be an atlas of C. Recall that this means that for $i, j \in I$ the function

$$\psi_{ji} = \varphi_j^{-1} \circ \varphi_i : U_{ij} = \varphi_i^{-1}(W_i \cap W_j) \to \varphi_j^{-1}(W_i \cap W_j) = U_{ji}$$

is a biholomorphic map. Now a differential form ω on C is given by a collection $(\omega_i = f_i(z)dz)_{i\in I}$ of differential forms $f_i(z)dz$ on U_i which are compatible. This compatibility means exactly that $\psi_{ji}^*\omega_j = \omega_i|_{U_{ij}}$. Here the pullback is defined by

$$\psi_{ji}^*\omega_j = \psi_{ji}^*(f_j(z)dz) = f_j(\psi_{ji}(z))d\psi_{ji}(z) = f_j(\psi_{ji}(z))\frac{d\psi_{ji}}{dz}dz.$$

Denote the $\mathbb{C}$-vector space of holomorphic differential forms on C by $H^0(C, \Omega_C)$.

i) The curve $C = \mathbb{CP}^1$ is covered by two charts

$$\varphi_1 : U_1 = \mathbb{C} \to \mathbb{CP}^1, z \mapsto [1 : z],$$
$$\varphi_2 : U_2 = \mathbb{C} \to \mathbb{CP}^1, w \mapsto [w : 1].$$

Use the definition above to show that $H^0(C, \Omega_C) = 0$.

ii) Given a holomorphic map $g : C \to C'$ of complex manifolds of dimension 1, describe how to define the pullback $g^*\omega$ of a holomorphic differential form ω on C' to C.

iii) Use part ii) to show that for $C = \mathbb{C}/\Lambda$ an elliptic curve ($\Lambda \subset \mathbb{C}$ a lattice), the space $H^0(C, \Omega_C)$ has dimension 1.

iv)* The definition of holomorphic differential forms (and their pullback) generalizes to higher dimensional complex manifolds. Show that for a lattice Λ inside $\mathbb{C}^g$, the complex manifold $T = \mathbb{C}^g/\Lambda$ has a g-dimensional space of holomorphic differentials and identify a basis of this space. Use this basis together with parts i) and ii) to show that any holomorphic map $\sigma : \mathbb{CP}^1 \to T$ is constant.

Due March 16.

Exercises with * are possibly harder and should be considered as optional challenges.