Algebraic Curves (FS 2017)

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Exercise Sheet 4

1. Consider the cover $U_i = \{[Z_0 : \ldots : Z_n] : Z_i \neq 0\}$ of projective space $\mathbb{P}^n$ with coordinates

$$\frac{Z_0}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_n}{Z_i}$$

on $U_i \cong \mathbb{C}^n$. With respect to this cover, the line bundle $\mathcal{O}(1)$ has the transition functions $\phi_{ij} = Z_j/Z_i$ on $U_i \cap U_j$. Here we mean that functions s_i on U_i are compatible (and define a section s of $\mathcal{O}(1)$) if $s_i = \phi_{ij} s_j$.

a) The line bundle $\mathcal{O}(d)$, $d \in \mathbb{Z}$, is defined as $\mathcal{O}(1)^{\otimes d}$ for $d \geq 1$ and $(\mathcal{O}(1)^*)^{\otimes -d}$ for $d < 0$, where $\mathcal{O}(1)^*$ is the dual bundle. What are its transition functions?

b) Show that $H^0(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $d < 0$.

c) Show that $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is isomorphic to the space of homogeneous polynomials in $Z_0, \ldots, Z_n$ of degree d. This has dimension $\binom{n+d}{n}$.

d)* The tautological line bundle $\mathcal{L}$ on $\mathbb{P}^n$ was defined as

$$\pi : \mathcal{L} = \{(v, l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n : v \in l\} \to \mathbb{P}^n, (v, l) \mapsto l.$$

Show that $\mathcal{L} \cong \mathcal{O}(-1)$.

2. a) Let C be a curve of genus 0. Show that C is isomorphic to $\mathbb{P}^1$. (*Hint*: To specify an isomorphism $C \to \mathbb{P}^1$ you have to give a meromorphic function on C.)

b) For the point $[1 : 0] \in \mathbb{P}^1$, show that the line bundles $\mathcal{O}([1 : 0])$ and $\mathcal{O}(1)$ are isomorphic by looking at their transition functions for the usual cover U_0, U_1. Conclude that $c_1(\mathcal{O}(n)) = n$.

c) Show that $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ with generator $\mathcal{O}(1)$ corresponding to $1 \in \mathbb{Z}$. Conclude that given any points $p_1, \ldots, p_r \in \mathbb{P}^1$ and $a_1, \ldots, a_r \in \mathbb{Z}$ one has $\mathcal{O}(a_1 p_1 + \ldots + a_r p_r) \cong \mathcal{O}(a_1 + \ldots + a_r)$.

3. Let C be a smooth curve and $\mathcal{L}$ a line bundle on C.

a) Show that $\mathcal{L}$ is isomorphic to the trivial line bundle $\mathcal{O}$ iff it has a section $s \in H^0(C, \mathcal{L})$ which vanishes nowhere.

b) For a nonzero meromorphic section s of $\mathcal{L}$ (i.e. a section of $\mathcal{L}$ on a set $U = C \setminus \{p_1, \ldots, p_n\}$) make sure you understand what is meant by its divisor $D = \mathrm{div}(s) \in \mathrm{Div}(C)$. For a second line bundle $\mathcal{L}'$ with nonzero meromorphic section s', we have a meromorphic section $s \otimes s'$ of $\mathcal{L} \otimes \mathcal{L}'$. Show that $\mathrm{div}(s \otimes s') = \mathrm{div}(s) + \mathrm{div}(s')$. Show also that a nonzero meromorphic section s with $\mathrm{div}(s) \geq 0$ (in the sense that all coefficients which appear are nonnegative) extends to a global section on all of C.

c) Show that for a divisor $D = \sum_i a_i p_i$ on C the line bundle $\mathcal{O}(D)$ has a meromorphic section s_D with $\operatorname{div}(s_D) = D$. (*Hint*: Under identifying $\mathcal{O}(D)(U)$ with meromorphic functions on U with zeroes and poles restricted by D, the section s_D corresponds to the function 1.)

d) For a nonzero meromorphic section s of $\mathcal{L}$ as in b) with $D = \operatorname{div}(s)$, show that $\mathcal{L} = \mathcal{O}(D)$. (*Hint*: Show that $\mathcal{L} \otimes \mathcal{O}(-D)$ is trivial by giving a global section that vanishes nowhere.)

e) For $c_1(\mathcal{L}) < 0$ show that $H^0(C, \mathcal{L}) = 0$.

4. Let $\Lambda = \langle v, w \rangle \subset \mathbb{C}$ be a lattice, then the elliptic curve $E = \mathbb{C}/\Lambda$ has $\omega = dz$ as a basis of $H^1(E, \Omega^1)$ and the cycles $a : [0,1] \to E, t \mapsto tv$ and $b : [0,1] \to E, t \mapsto tw$ as a basis of $H_1(E, \mathbb{Z})$. With respect to these choices show that $\operatorname{Jac}(E) = \mathbb{C}/H_1(E, \mathbb{Z})$ is canonically isomorphic to E and compute the Abel-Jacobi map

$$\mathrm{AJ} : \operatorname{Div}^0(E) \to \operatorname{Jac}(E).$$

5. In the lecture you saw that morphisms $f : X \to \mathbb{P}^n$ correspond bijectively to the data of a line bundle $\mathcal{L}$ on X together with sections $s_0, \dots, s_n$ of $\mathcal{L}$ not vanishing simultaneously. For the following maps f give the corresponding line bundle $\mathcal{L}$ and describe the sections s_i. (*Note*: Sometimes it is not easy to describe the sections s_i explicitly, but (except when indicated) you can describe $\operatorname{div}(s_i)$, which determines s_i up to scaling.)

a) $f : \mathbb{P}^1 \to \mathbb{P}^3, [s : t] \mapsto [s^3 : s^2 t : s t^2 : t^3]$

b) $f : C \to \mathbb{P}^1$ interpreted as a meromorphic function f on C

c) $\wp : \mathbb{C}/\Lambda \to \mathbb{P}^1$ the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right] \quad \text{for } z \in \mathbb{C} \setminus \Lambda.$$

Note: Giving s_1 here explicitly in terms of the lattice Λ is quite non-trivial and not part of this exercise!

d) $\wp' : \mathbb{C}/\Lambda \to \mathbb{P}^1$

e) $f : \mathbb{C}/\Lambda \to \mathbb{P}^2, z \mapsto [\wp(z) : \wp'(z) : 1]$ for $z \neq 0$ (*Remark*: As we have seen on Sheet 1, f extends to a function on all of $\mathbb{C}/\Lambda$.)

f)* $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, ([s : t], [u : v]) \mapsto [su : sv : tu : tv]$

Due May 9.

Exercises with * are possibly harder and should be considered as optional challenges.