Prof. R. Pandharipande
J. Schmitt

Exercise Sheet 1 - Solutions

1. Show that every regular (holomorphic) function on a nonsingular, connected projective curve C is constant. Can you write down two proofs, once seeing C as an algebraic variety and once as a complex manifold?

Solution

Algebra style: A regular function on C corresponds to a morphism $f : C \to \mathbb{C}$ of algebraic varieties. As C is projective, its image under any algebraic morphism $g : C \to V$ to some variety V is closed in V and itself projective. In particular this holds for $f(C) \subset \mathbb{C}$. On the other hand, subvarieties of $\mathbb{C}$ are the zero-sets of ideals $I \subset \mathbb{C}[x]$ inside the ring of regular functions $\mathbb{C}[x]$ on $\mathbb{C}$. This ring is a principal ideal domain, so $I = (h)$ for $h \in \mathbb{C}[x]$. If $h = 0$, we have for its vanishing set $V(h) = \mathbb{C}$, but $f(C) = V(h) \neq \mathbb{C}$ since $\mathbb{C}$ is not projective. If $h \neq 0$ then it has only finitely many zeroes, so $V(h) = f(C)$ is a finite collection of points. As C is connected, so is $f(C)$, hence $f(C)$ is a single point, i.e. f is constant.

Analysis style: As $f : C \to \mathbb{C}$ is continuous (with respect to the complex topology on C and $\mathbb{C}$) and as C is compact, $f(C)$ is compact too. Let $p \in C$ be a point where $|f|$ obtains its maximum, then taking a little chart $\varphi : \mathbb{C} \supset U \to C$ around p we find that the holomorphic function $f \circ \varphi$ obtains a maximum of its absolute value at $\varphi^{-1}(p)$, hence is constant equal to some $c \in \mathbb{C}$ by the maximum principle. But the set of points $q \in C$ such that f is constant equal to c in a neighbourhood of q is open (by definition) and closed (essentially by the identity theorem). Thus as C is connected, the function f is globally equal to c.

2. Consider the meromorphic function $f(z) = \frac{z^3-z^2}{z^2+1}$ on $\mathbb{C}$.

i) Using the identification $\mathbb{C} \subset \mathbb{CP}^1$, $z \mapsto [1 : z]$, on source and target, identify f as a map $\mathbb{CP}^1 \to \mathbb{CP}^1$ and write it in the form $[x : y] \mapsto [F(x, y), G(x, y)]$ for homogeneous polynomials F, G of the same degree, having no common zeroes on $\mathbb{CP}^1$.

ii) Compute $\mathrm{div}(f)$.

Solution

i) Using the identification given above, we want F, G to satisfy

$$[x : y] = [1 : z] \mapsto [F(x, y) : G(x, y)] = [1 : \frac{z^3 - z^2}{z^2 + 1}]$$

for $x \neq 0$ and $F(x,y) \neq 0$. The first equality gives $z = y/x$ and then the second equality reads

$$\frac{G(x,y)}{F(x,y)} = \frac{(y/x)^3 - (y/x)^2}{(y/x)^2 + 1} = \frac{\frac{y^3 - y^2 x}{x^3}}{\frac{y^2+x^2}{x^2}} = \frac{y^3 - y^2 x}{x(y^2 + x^2)},$$

so we can take $G(x,y) = y^3 - y^2 x$, $F(x,y) = x(y^2 + x^2)$. We see that we have essentially just homogenized the numerator and denominator of the original rational function, adding a suitable power of x in the denominator as they had different degrees in z.

ii) We have $y^3 - y^2 x = y^2(y - x)$ and $x(y^2 + x^2) = x(y + ix)(y - ix)$, so the zeroes of the numerator on $\mathbb{CP}^1$ are at $y = 0$ (i.e. $[1:0]$) with multiplicity 2 and at $y - x = 0$ (i.e. $[1:1]$). The zeroes of the denominator are at $[0:1]$, $[1:-i]$ and $[1:i]$ (all simple), so

$$\operatorname{div}(f) = 2[1:0] + [1:1] - [0:1] - [1:-i] - [1:i].$$

3. Let $\Lambda \subset \mathbb{C}$ be a lattice (i.e. $\Lambda = \mathbb{Z}v + \mathbb{Z}w$ for $v, w \in \mathbb{C}$ two $\mathbb{R}$-linearly independent vectors).

i) Convince yourself that $\mathbb{C}/\Lambda$ with the quotient topology is a torus (i.e. a compact oriented surface of genus 1) and that it has the structure of a complex manifold of dimension 1, i.e. an algebraic curve.

ii) A meromorphic function f on $\mathbb{C}$ is called *doubly periodic* (with respect to Λ) if $f(z + \omega) = f(z)$ for all $\omega \in \Lambda$. Show that every holomorphic doubly periodic function $f : \mathbb{C} \to \mathbb{C}$ is constant.

iii) Assume f is doubly periodic with no poles on the boundary of the *fundamental domain*

$$D = \{\lambda v + \mu w : 0 \leq \lambda, \mu \leq 1\}.$$

Show that the sum of the residues of all poles of f inside D is zero. Conclude that no such f can have a unique simple pole in D.

iv) The Weierstrass $\wp$-function of the lattice Λ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right] \quad \text{for } z \in \mathbb{C} \setminus \Lambda.$$

With some work you can show that this series converges absolutely for every $z \in \mathbb{C} \setminus \Lambda$ and uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$, so $\wp$ defines a meromorphic function with double poles at the elements of Λ.

Show that the derivative $\wp'$ is doubly periodic and odd ($\wp'(-z) = -\wp'(z)$) and that $\wp$ is even ($\wp(-z) = \wp(z)$). Use this to show that $\wp$ is doubly periodic.

v) Show that $\wp'$ has simple zeroes exactly at the points $\frac{v}{2} + \Lambda$, $\frac{w}{2} + \Lambda$, $\frac{v+w}{2} + \Lambda$. *Hint:* Here you can use that $\wp' : \mathbb{C}/\Lambda \dashrightarrow \mathbb{C}$ satisfies $\deg(\operatorname{div}(\wp')) = 0$.

vi) Prove that every doubly periodic function f with double poles exactly at the elements of Λ is of the form $f(z) = a\wp(z) + b$ for $a \in \mathbb{C}^*, b \in \mathbb{C}$. So up to translation ($z \mapsto z + z_0$), scaling and adding constant functions, $\wp$ is the unique doubly periodic function having as poles the Λ-translates of one double pole!

vii) Let the Laurent expansion of (the even function) $\wp$ around $z = 0$ be given by

$$\wp(z) = \frac{1}{z^2} + c + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^4 + O(z^6).$$

Show that $c = 0$ and that $\wp$ satisfies the differential equation

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp(z) - g_3.$$

Hint: Use part ii).

Thus we have a holomorphic map

$$g : \mathbb{C}/\Lambda \to E = V(Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3) \subset \mathbb{CP}^2, z \mapsto [\wp(z) : \wp'(z) : 1],$$

with coordinates $[X : Y : Z]$ on $\mathbb{CP}^2$, sending $z = 0$ to $[0 : 1 : 0]$.

viii)* Show that E is a smooth algebraic curve if $(g_2)^3 - 27(g_3)^2 \neq 0$ (if you are unfamiliar with such calculations in projective space, just show that $V(y^2 - 4x^3 + g_2x + g_3) \subset \mathbb{C}^2$ is smooth in this case).

ix)* Conclude that g is an isomorphism.

Solution

i) The first part is basic topology. To obtain an atlas of $\mathbb{C}/\Lambda$ as a complex manifold, one can restrict the quotient map $\mathbb{C} \to \mathbb{C}/\Lambda$ to small open subsets $U \subset \mathbb{C}$ (in the sense that $U \cap (\cup_{\lambda \in \Lambda \setminus \{0\}} \lambda + U) = \emptyset$. These maps $U \to \mathbb{C}/\Lambda$ cover $\mathbb{C}/\Lambda$ and the transition maps are simply translations $z \mapsto z + \lambda$ for $\lambda \in \Lambda$, so in particular holomorphic.

ii) With definition of D as in iii) we see $f(\mathbb{C}) = f(D)$ by double periodicity, which is compact (as D compact, f continuous) and thus bounded. By Liouville's theorem, every bounded holomorphic function is constant.

iii) By the residue theorem we have

$$2\pi i \sum_{p \in \mathrm{Int}(D)} \mathrm{Res}_p f = \int_{\partial D} f(z)dz.$$

The path ∂D decomposes into line segments $0 \to v, v \to v + w, v + w \to w, w \to 0$ and by double periodicity, the integrals $0 \to v$ and $v + w \to w$ cancel each other (similarly $v \to v + w$ and $w \to 0$), hence the integral on the right vanishes.

If f had a single simple pole p, we would have $\mathrm{Res}_p f = 0$ giving a contradiction to the fact that f has a pole at p, so no such f can exist.

iv) As the series of holomorphic functions converges uniformly on compact subsets, we can switch differentiation and summation and see

$$\wp'(z) = -\frac{2}{z^3} + \sum_{w\in\Lambda\setminus\{0\}} -\frac{2}{(z-w)^3} = \sum_{w\in\Lambda} -\frac{2}{(z-w)^3}.$$

The sum on the right is clearly invariant under shifts $z \mapsto z + w_0$ by elements of $w_0 \in \Lambda$ (this just induces a bijection of the index set Λ of the summation), so $\wp'$ is doubly periodic. Similarly, the bijection $\Lambda \to \Lambda, w \mapsto -w$ is used to show that $\wp'$ is odd. The same bijection also works to show that $\wp$ is even (using $1/(-w)^2 = 1/w^2$).

For $\wp$ the double periodicity is not entirely clear a priori because of the summands $-1/(w^2)$ we added to achieve convergence. However, given the generator $v \in \Lambda$, the function $z \mapsto \wp(z+v)-\wp(z)$ has derivative $\wp'(z+v)-\wp'(z) = 0$, so as $\mathbb{C} \setminus \Lambda$ is connected, it is constant. Its value is determined to be zero by evaluation at $z = -v/2 \notin \Lambda$ (where $\wp$ even is used for the second equality):

$$\wp(-v/2+v) = \wp(v/2) = \wp(-v/2) \implies \wp(z+v) - \wp(z)|_{z=-v/2} = 0.$$

The same argument shows periodicity of $\wp$ with v replaced by w, so indeed $\wp$ is doubly periodic.

v) The meromorphic function $\wp' : \mathbb{C}/\Lambda \dashrightarrow \mathbb{C}$ gives a holomorphic function $\wp' : \mathbb{C}/\Lambda \to \mathbb{CP}^1$. The preimage of ∞ is the set of poles of $\wp'$ and from the sum above we see that it has exactly one pole of order 3 at the class $[0] \in \mathbb{C}/\Lambda$. If $z_1, \ldots, z_r$ are the zeroes of $\wp'$ with multiplicities $n_1, \ldots, n_r$, then

$$\deg(\mathrm{div}(\wp')) = \deg(-3[\infty] + n_1[z_1] + \ldots + n_r[z_r]) = -3 + n_1 + \ldots + n_r = 0.$$

Thus $\wp'$ can have at most $r = 3$ zeroes (all $n_i \geq 1$) with $n_i = 1$ for $r = 3$. But for $2z \in \Lambda$ and $z \notin \Lambda$ we have $\wp'(z) = \wp'(z-2z) = \wp'(-z) = -\wp'(z)$, using that $\wp'$ is odd. Hence $\wp'(z) = 0$ for $z = \frac{v}{2}, \frac{w}{2}, \frac{v+w}{2}$. We have thus found three distinct zeroes and by the argument above they are all simple and there are no further zeroes.

vi) If f has double poles exactly at the elements of Λ, we can write the Laurent expansion of f around $z = 0$ as

$$f(z) = \frac{a}{z^2} + O(z^{-1}).$$

On the other hand, the Laurent expansion of $\wp(z)$ is given by

$$\wp(z) = \frac{1}{z^2} + O(z^{-1}).$$

The function $f(z) - a\wp(z)$ is doubly periodic with at most a single simple pole at $z = 0$, thus constant equal to some $b \in \mathbb{C}$ by ii).

vii) We have $c = 0$ by evaluating the sum over $w \in \Lambda \setminus \{0\}$ in the definition of $\wp$ at $z = 0$. The Laurent expansion of $\wp'(z)$ at $z = 0$ is given by

$$\wp'(z) = -\frac{2}{z^3} + \frac{1}{10}g_2 z + \frac{1}{7}g_3 z^3 + O(z^5).$$

By the Cauchy product rule for Laurent series, its square is given by

$$[\wp'(z)]^2 = \frac{4}{z^6} - \frac{2}{5}g_2 z^{-2} - \frac{4}{7}g_3 + O(z).$$

Similarly, the cube of $\wp(z)$ is given by

$$[\wp(z)]^3 = \frac{1}{z^6} + \frac{3}{20}g_2 z^{-2} + \frac{3}{28}g_3 + O(z).$$

Summing up we find

$$\begin{aligned}&[\wp'(z)]^2 - 4[\wp(z)]^3 + g_2\wp(z) + g_3\\ =&(4-4)z^{-6} + (-\frac{2}{5} - \frac{3}{5} + 1)g_2 z^{-2} + (-\frac{4}{7} - \frac{3}{7} + 1)g_3 + O(z) = O(z)\end{aligned}$$

Thus the doubly periodic function $[\wp'(z)]^2 - 4[\wp(z)]^3 + g_2\wp(z) + g_3$ is holomorphic and vanishes at zero, hence is equal to zero by ii).

The map g sends $z = 0$ to $[0 : 1 : 0]$ as $\wp'$ has a triple pole and $\wp$ has only a double pole at $z = 0$, so roughly

$$[\wp(z) : \wp'(z) : 1] = [\wp(z)/\wp'(z) : 1 : 1/\wp'(z)] \approx [z : 1 : z^3] \xrightarrow{z\to 0} [0 : 1 : 0].$$

viii)* For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ the inverse function theorem from Analysis/Differential geometry states that the level set $\{f = 0\}$ is a smooth manifold of dimension $n - 1$ around a point $p \in \{f = 0\}$ if the gradient

$$\operatorname{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

is nonzero at p. The same is true if we replace $\mathbb{R}$ by $\mathbb{C}$ and "smooth" by "complex".

To apply this to our setting, we can check that E is smooth (i.e. a complex manifold) on an open cover of E. One such open set is given by the standard chart

$$\mathbb{C}^2 \hookrightarrow \mathbb{CP}^2, (x, y) \mapsto [x : y : 1].$$

When looking at the equation of E, this just means we set $Z = 1$, so as described in the hint, we want to prove smoothness of $V(f = y^2 - 4x^3 + g_2 x + g_3) \subset \mathbb{C}^2$.

The partial derivatives of f are given by

$$\frac{\partial f}{\partial x} = -12x^2 + g_2, \frac{\partial f}{\partial y} = 2y.$$

For them to simultaneously vanish at a point of E, we need to have $y = 0, -12x^2 + g_2 = 0$ and of course still the original equation $y^2 - 4x^3 + g_2 x +$

$g_3 = 0$ of E. Inserting the solutions $(x, y) = (\pm\sqrt{g_2/12}, 0)$ of the first two equations into the second yields

$$0 = \pm(-4\sqrt{g_2/12}^3 + g_2\sqrt{g_2/12}) + g_3 = \pm\frac{1}{3}g_2\sqrt{g_2/3} + g_3$$

But $g_3 = \mp 1/3 g_2\sqrt{g_2/3}$ is equivalent (by taking the square) to $(g_3)^2 = 1/27(g_2)^3$, so there exists a non-smooth point in $\mathbb{C}^2 \cap E$ iff this equation is satisfied.

Now either one can check smoothness on the other two charts of $\mathbb{CP}^2$ by hand, or one uses that in general (by a slightly more careful analysis of the three charts) we can find the non-smooth points of $V(F(X, Y, Z)) \subset \mathbb{CP}^2$ as the simultaneous vanishing locus of

$$\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}, F\right).$$

In our case this gives

$$(-12X^2 + g_2Z^2, 2YZ, Y^2 + 2g_2XZ + 3g_3Z^2, Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3).$$

Our argument above shows that for $Z \neq 0$ we have a solution iff $(g_2)^3 - 27(g_3)^2 = 0$, because for any solution (X, Y, Z) also $\lambda(X, Y, Z)$ is a solution, so we can scale Z to be 1. For $Z = 0$ the equations simplify to

$$(-12X^2, 0, Y^2, -4X^3)$$

implying that $X = 0, Y = 0, Z = 0$, which gives a contradiction. So we see that the points of $E \cap \{Z = 0\}$ are always smooth. This finishes the proof that E is smooth iff $(g_2)^3 - 27(g_3)^2 \neq 0$.

Now the lecture defines an algebraic curve to be connected, so this is something we still have to check for E. The easiest argument uses the Theorem of Bézout: assume E decomposes into two components E_1, E_2 of degrees e_1, e_2 (in the sense that E_1 is cut out by a polynomial F_1 of degree e_1 and E_2 by a polynomial F_2 of degree e_2). For their defining polynomials it follows then that (after possibly scaling by a nonzero constant) $F_1F_2 = Y^2Z - 4X^3 + g_2XZ^2 + g_3Z^3 = F$. By Bézout's theorem the curves E_1, E_2 must meet in e_1e_2 points (counted with multiplicity). All we need from this is that they meet in some point $p \in E$. At this point, we have $F_1(p) = F_2(p) = 0$, so we see

$$\frac{\partial F}{\partial X_i}(p) = \frac{\partial F_1}{\partial X_i}(p)F_2(p) + F_1(p)\frac{\partial F_2}{\partial X_i} = 0 + 0 = 0$$

for $X_i = X, Y, Z$. In other words the point p is a singular (i.e. non-smooth) point of E. This is a contradiction to the computation above, so E can only have one component and thus it is connected.

ix)* By part i) we have that $\mathbb{C}/\Lambda$ is an algebraic curve and by part viii) we know that E is an algebraic curve. As $\wp(z)$ is non-constant, we see that g is a non-constant map between these curves. To show that g is an isomorphism we can for instance show that it is of degree 1 (i.e. every point has a unique preimage point, so g is surjective and injective). But to compute the degree we can simply check the number of preimages for any point in E. Looking at $p = [0 : 1 : 0] \in E$ we know this point can only have $z = 0 \in \mathbb{C}/\Lambda$ as preimage (as reaching it via g requires $\wp'(z)$ to have a pole). To compute the multiplicity of the preimage, we look at the chart $Y = 1$ with coordinates $x = X/Y, z = Z/Y$, so g is given in these coordinates by $(x, y) = (\wp(z)/\wp'(z), 1/\wp'(z))$. The curve E is cut out in this chart by $z - 4x^3 + g_2xz^2 + g_3z^3$. We see that the partial derivative of this equation with respect to z does not vanish at $(x, z) = (0, 0)$, so around $(0, 0)$ the curve E is the graph of a function $z(x)$. In other words, in a small neighbourhood of $0 \in \mathbb{C}$ we have a chart $x \mapsto (x, z(x))$ of E with inverse given by $\pi : (x, z) \mapsto x$.

To compute the multiplicity of the preimage $z = 0$ of $p \in E$ under g we use this chart and see that $g^{-1}(p) = g^{-1}(\pi^{-1}(\{x = 0\}))$, so we only have to see with which multiplicity $x = X/Y = \wp(z)/\wp'(z)$ vanishes at $z = 0$. Since $\wp(z)$ has a double pole at $z = 0$ and $\wp'(z)$ has a triple pole, this multiplicity is 1 and this is also the degree of g, so g is an isomorphism.

Due March 2.

Exercises with * are possibly harder and should be considered as optional challenges.