Prof. R. Pandharipande
J. Schmitt

Exercise Sheet 2 - Solutions

1. Show that for curves C_g, C_h of genus g, h respectively there does not exist a non-constant holomorphic map $f : C_g \to C_h$ if $g < h$. *Hint:* Given the results from the lecture, the proof is very short, so don't try something complicated!

Solution Assume such an f exists and let $d > 0$ be its degree, then Riemann-Hurwitz tells us that

$$2g - 2 = d(2h - 2) + b$$

for $b \geq 0$ the sum of the ramification indices of f. As $0 \leq g < h$, we have $h \geq 1$ so $2h - 2 \geq 0$ and we have the following chain of inequalities

$$2g - 2 < 2h - 2 \leq d(2h - 2) \leq d(2h - 2) + b,$$

giving a contradiction to the equality above.

2. In the lecture you saw the following (seeming) contradiction to the Riemann-Hurwitz formula: given a function $f : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ with $f^{-1}([\infty]) = d[0]$, it looks like $f(z) = \frac{c}{z^d} + \ldots$ around $z = 0$, so $f'(z) = \frac{-cd}{z^{d+1}} + \ldots$. Computing the divisor we have

$$\operatorname{div}(f') = [p_1] + \ldots [p_b] - (d + 1)[0].$$

As this should have degree 0 we know that (counted with multiplicities) f' has $b = d + 1$ zeroes, which are the ramification points of f. However, as the genus of $\mathbb{C}/\Lambda$ is 1 and the genus of $\mathbb{CP}^1$ is 0, Riemann-Hurwitz tells us that $2 - 2 = d(-2) + b$, so we expect a total of $b = 2d$ ramification points. What is wrong in this counter-example?

Solution The equation $f'(z) = 0$ tells us all ramification points that do not map to ∞. However, ∞ is actually a branch point of f as $f^{-1}(\infty) = d[0]$ and the ramification index is $d - 1$, so the total ramification index is indeed $d + 1 + (d - 1) = 2d$.

3. In this exercise we want to recall the notion of a holomorphic differential form. Let C be a complex manifold of dimension 1 and let $(\varphi_i : U_i \xrightarrow{\sim} W_i \subset C)_{i \in I}$ be an atlas of C. Recall that this means that for $i, j \in I$ the function

$$\psi_{ji} = \varphi_j^{-1} \circ \varphi_i : U_{ij} = \varphi_i^{-1}(W_i \cap W_j) \to \varphi_j^{-1}(W_i \cap W_j) = U_{ji}$$

is a biholomorphic map. Now a differential form ω on C is given by a collection $(\omega_i = f_i(z)dz)_{i \in I}$ of differential forms $f_i(z)dz$ on U_i which are compatible. This compatibility means exactly that $\psi_{ji}^* \omega_j = \omega_i|_{U_{ij}}$. Here the pullback is defined by

$$\psi_{ji}^* \omega_j = \psi_{ji}^*(f_j(z)dz) = f_j(\psi_{ji}(z))d\psi_{ji}(z) = f_j(\psi_{ji}(z))\frac{d\psi_{ji}}{dz}dz.$$

Denote the $\mathbb{C}$-vector space of holomorphic differential forms on C by $H^0(C, \Omega_C)$.

i) The curve $C = \mathbb{CP}^1$ is covered by two charts

$$\varphi_1 : U_1 = \mathbb{C} \to \mathbb{CP}^1, z \mapsto [1 : z],$$
$$\varphi_2 : U_2 = \mathbb{C} \to \mathbb{CP}^1, w \mapsto [w : 1].$$

Use the definition above to show that $H^0(C, \Omega_C) = 0$.

ii) Given a holomorphic map $g : C \to C'$ of complex manifolds of dimension 1, describe how to define the pullback $g^*\omega$ of a holomorphic differential form ω on C' to C.

iii) Use part ii) to show that for $C = \mathbb{C}/\Lambda$ an elliptic curve ($\Lambda \subset \mathbb{C}$ a lattice), the space $H^0(C, \Omega_C)$ has dimension 1.

iv)* The definition of holomorphic differential forms (and their pullback) generalizes to higher dimensional complex manifolds. Show that for a lattice Λ inside $\mathbb{C}^g$, the complex manifold $T = \mathbb{C}^g/\Lambda$ has a g-dimensional space of holomorphic differentials and identify a basis of this space. Use this basis together with parts i) and ii) to show that any holomorphic map $\sigma : \mathbb{CP}^1 \to T$ is constant.

Solution

i) Let $\omega_1 = f(z)dz$ and $\omega_2 = g(w)dw$. The base change map $\psi_{21} : U_1 \supset \mathbb{C}^* \to \mathbb{C}^* \subset U_2$ is given by $\psi_{21}(z) = 1/z$. Indeed we have

$$\psi_{21} = \varphi_2^{-1}(\varphi_1(z)) = \varphi_2^{-1}([1 : z]) = \varphi_2^{-1}([1/z : 1]) = 1/z.$$

Then the condition on the differentials ω_1, ω_2 is

$$f(z)dz = \omega_1 = \psi_{21}^*\omega_2 = g(\psi_{21}(z))d\psi_{21}(z) = g(1/z)\frac{-dz}{z^2}.$$

Now f and g are holomorphic functions on all of $\mathbb{C}$, so they are described entirely by their power series expansion around zero. But then looking at the equation $f(z) = -g(1/z)/z^2$ we see that only $f = g = 0$ are a solution, as the left hand side is holomorphic at $z = 0$ and the right is not unless $g = 0$. Thus $\omega_1 = 0, \omega_2 = 0$, so every holomorphic differential vanishes on $\mathbb{CP}^1$.

ii) Let $g : C \to C'$ be holomorphic, let ω be a holomorphic differential on C', let $\varphi' : U' \to W' \subset C'$ be a holomorphic chart such that ω is given on U' by $f(z)dz$. Let $\varphi : U \to W \subset C$ be a chart of C such that $g(W) \subset W'$, then as g is holomorphic, we have that the composition $(\varphi')^{-1} \circ g \circ \varphi : U \to U'$ is holomorphic. Then the differential $g^*\omega$ on C is given in the chart φ by $((\varphi')^{-1} \circ g \circ \varphi)^* f(z)dz$. As C is covered by such charts φ (as the charts φ' cover C'), this uniquely describes $g^*\omega$. One shows that the differential forms on the various charts U are compatible by using the complex chain rule.

iii) We have a holomorphic map $g : \mathbb{C} \to C = \mathbb{C}/\Lambda$ given by the quotient map. If ω is a differential on C, its pullback $g^*\omega$ is a differential on $\mathbb{C}$, so it is of the form $g^*\omega = f(z)dz$ for f a holomorphic function on $\mathbb{C}$. Now as

g is invariant under translation on the source by elements of Λ, the form $g^*\omega$ (and thus f) is doubly periodic. But a holomorphic doubly periodic function f is constant, so $g^*\omega = c \cdot dz$ for some $c \in \mathbb{C}$. By restricting g to small subsets of $\mathbb{C}$ we obtain an atlas of C, so indeed the differential form ω is just given by $c \cdot dz$ in all such charts. Hence the map

$$\mathbb{C} \to H^0(C, \Omega_C), c \mapsto c \cdot dz$$

is an isomorphism.

iv)* Repeating the argument from iii) with the map $h : \mathbb{C}^g \to \mathbb{C}^g/\Lambda$ we see that the pullback $h^*\omega$ of a holomorphic differential ω on $\mathbb{C}^g/\Lambda$ is given by $f_1(z)dz_1 + \ldots + f_g(z)dz_g$. All functions f_i are invariant under shifts by Λ, so as $\mathbb{C}^g/\Lambda$ is compact, by the maximum principle they are constant. Thus the space of differential forms is given by $\omega = c_1dz_1 + \ldots + c_gdz_g$ for $(c_1, \ldots, c_g) \in \mathbb{C}^g$, so g-dimensional.

Assume $\sigma : \mathbb{CP}^1 \to T$ is holomorphic with $p \in \mathbb{CP}^1$ mapping to the image of some chart $\varphi : \mathbb{C}^g \supset U \to \mathbb{C}^g/\lambda$, given by a restriction of the quotient map h. Then around p, the function $\varphi^{-1} \circ f$ is given by its coordinate functions $(\sigma_1(z), \ldots, \sigma_g(z))$. For each $i = 1, \ldots, g$ the differential form dz_i on T pulls back to the zero form on $\mathbb{CP}^1$, as all holomorphic differentials on $\mathbb{CP}^1$ vanish! Writing this explicitly we have

$$0 = \sigma^* dz_i = d(z_i \circ \sigma) = d\sigma_i = \frac{d\sigma_i}{dz} dz.$$

Thus the derivative of σ_i is zero, so σ_i is constant for all i. Hence σ is constant as claimed above.

Due March 16.

Exercises with * are possibly harder and should be considered as optional challenges.