Algebraic Curves (FS 2017)

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Exercise Sheet 4 - Solutions

1. Consider the cover $U_i = \{[Z_0 : \ldots : Z_n] : Z_i \neq 0\}$ of projective space $\mathbb{P}^n$ with coordinates

$$\frac{Z_0}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_n}{Z_i}$$

on $U_i \cong \mathbb{C}^n$. With respect to this cover, the line bundle $\mathcal{O}(1)$ has the transition functions $\phi_{ij} = Z_j/Z_i$ on $U_i \cap U_j$. Here we mean that functions s_i on U_i are compatible (and define a section s of $\mathcal{O}(1)$) if $s_i = \phi_{ij} s_j$.

a) The line bundle $\mathcal{O}(d)$, $d \in \mathbb{Z}$, is defined as $\mathcal{O}(1)^{\otimes d}$ for $d \geq 1$ and $(\mathcal{O}(1)^*)^{\otimes -d}$ for $d < 0$, where $\mathcal{O}(1)^*$ is the dual bundle. What are its transition functions?

b) Show that $H^0(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $d < 0$.

c) Show that $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is isomorphic to the space of homogeneous polynomials in $Z_0, \ldots, Z_n$ of degree d. This has dimension $\binom{n+d}{n}$.

d)* The tautological line bundle $\mathcal{L}$ on $\mathbb{P}^n$ was defined as

$$\pi : \mathcal{L} = \{(v, l) \in \mathbb{C}^{n+1} \times \mathbb{P}^n : v \in l\} \to \mathbb{P}^n, (v, l) \mapsto l.$$

Show that $\mathcal{L} \cong \mathcal{O}(-1)$.

Solution

a) For $\mathcal{L}, \mathcal{L}'$ line bundles defined by transition functions ϕ_{ij}, ϕ'_{ij} with respect to some cover, their tensor product is defined by $\phi_{ij}\phi'_{ij}$ and the dual bundle $\mathcal{L}^*$ is defined by ϕ_{ij}^{-1}. This implies that the transition functions of $\mathcal{O}(d)$ are given by $\psi_{ij} = (\phi_{ij})^d = (Z_j/Z_i)^d$.

b) Assume a section $f \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ is given in the chart U_i by the polynomial

$$f_i \in \mathbb{C}\left[\frac{Z_0}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right].$$

Then to be compatible, we require that $f_i = (Z_j/Z_i)^d f_j$. Note that this equality is an equality of functions on $U_i \cap U_j$ and the ring of functions there is given by

$$\mathbb{C}\left[\frac{Z_0}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right]\left[\left(\frac{Z_j}{Z_i}\right)^{-1}\right].$$

This means in particular that this must be an equality of Laurent polynomials in $Z_0, \ldots, Z_n$.

Rewrite the desired equation as $f_i Z_i^d = Z_j^d f_j$, then as $d < 0$, we know that the left hand side has degree at most $d < 0$ in Z_i (remember f_i is a polynomial in terms Z_k/Z_i) and the right hand side has only nonnegative powers of Z_i appearing. This shows that both sides must be identically zero, so $f_i = 0$ for all i, i.e. $f = 0$.

c) The same arguments as in b) lead to the desired equality $f_i Z_i^d = Z_j^d f_j$. If f_i has terms of degree greater than d in the variables Z_k/Z_i, the left hand side has terms of strictly negative degree in Z_i and we have a contradiction as before. Conversely, if f is a polynomial of degree at most d, the term Z_i^d cancels all denominators on the left and $f_j = f_i(Z_i/Z_j)^d$ is a well defined polynomial in the variables Z_k/Z_d. One checks that for any such f_i the functions $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$ defined by this formula are compatible and thus give a section f of $\mathcal{O}(d)$. The homogeneous polynomial of degree d in $Z_0, \ldots, Z_d$ associated to f is exactly $f_i Z_i^d$.

By elementary enumerative combinatorics, the set of monomials in $Z_0, \ldots, Z_n$ of degree d has cardinality $\binom{n+d}{n}$.

d)* We saw in the class that the line bundle $\mathcal{L}$ is trivial on the open cover $U_0, \ldots, U_n$ of $\mathbb{P}^n$ and on U_i for a regular function

$$s_i(\frac{Z_0}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_n}{Z_i})$$

we obtain the section

$$U_i \to \pi^{-1}(U_i), P = [\frac{Z_0}{Z_i}, \ldots, 1, \ldots, \frac{Z_n}{Z_i}] \mapsto \left((s_i \frac{Z_0}{Z_i}, s_i \frac{Z_1}{Z_i}, \ldots, s_i, \ldots, s_i \frac{Z_n}{Z_i}), P \right).$$

We have a similar expression for s_j. Then s_i, s_j define compatible sections iff

$$(s_i \frac{Z_0}{Z_i}, s_i \frac{Z_1}{Z_i}, \ldots, s_i, \ldots, s_i \frac{Z_n}{Z_i}) = (s_j \frac{Z_0}{Z_j}, s_j \frac{Z_1}{Z_j}, \ldots, s_j, \ldots, s_j \frac{Z_n}{Z_j}). \tag{1}$$

Comparing the first coordinates, we see that necessarily $s_i Z_0/Z_i = s_j Z_0/Z_j$ or in other words $s_i = (Z_i/Z_j)s_j$. Conversely, one checks that under this assumption all other components in (1) agree as well. Thus the transition functions for $\mathcal{L}$ are $\phi'_{ij} = Z_i/Z_j = \phi_{ij}^{-1}$, where ϕ_{ij} are the transition functions of $\mathcal{O}(1)$ from above. Thus we see $\mathcal{L} = \mathcal{O}(1)^* = \mathcal{O}(-1)$.

2. a) Let C be a curve of genus 0. Show that C is isomorphic to $\mathbb{P}^1$. (*Hint*: To specify an isomorphism $C \to \mathbb{P}^1$ you have to give a meromorphic function on C.)

b) For the point $[1:0] \in \mathbb{P}^1$, show that the line bundles $\mathcal{O}([1:0])$ and $\mathcal{O}(1)$ are isomorphic by looking at their transition functions for the usual cover U_0, U_1. Conclude that $c_1(\mathcal{O}(n)) = n$.

c) Show that $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ with generator $\mathcal{O}(1)$ corresponding to $1 \in \mathbb{Z}$. Conclude that given any points $p_1, \ldots, p_r \in \mathbb{P}^1$ and $a_1, \ldots, a_r \in \mathbb{Z}$ one has $\mathcal{O}(a_1 p_1 + \ldots + a_r p_r) \cong \mathcal{O}(a_1 + \ldots + a_r)$.

Solution For the entire exercise it will be useful to recall the exact sequence involving the Jacobian in genus $g = 0$. Here the Jacobian is given by $\mathrm{Jac}(C) = \mathbb{C}^0/H_1(C,\mathbb{Z}) = \{0\}$, so the sequence reads

$$0 \to \mathbb{C}^* \to K(C)^* \xrightarrow{\mathrm{div}} \mathrm{Div}^0(C) \to 0.$$

On the other hand we saw that the Jacobian is the degree 0 part of the Picard group, i.e. the kernel of the degree map $c_1 : \mathrm{Pic}(C) \to \mathbb{Z}$. As the Jacobian is trivial, this map is injective (and trivially surjective), hence an isomorphism.

a) Let $p, q \in C$ be two distinct points, then $p - q \in \mathrm{Div}^0(C)$. By the exact sequence above there exists a meromorphic function f on C with $\mathrm{div}(f) = p - q$. This function defines a morphism $f : C \to \mathbb{P}^1$ and the preimage of $0 \in \mathbb{P}^1$ is exactly p (with multiplicity 1). Thus f has degree 1, so every point in $\mathbb{P}^1$ has exactly one preimage point. This implies that f is a bijection.

   To be super-precise, we still need to show that the inverse of f is also smooth/algebraic. In the algebraic world, it is true in general that a morphism of smooth, irreducible varieties over $\mathbb{C}$ which is bijective is automatically an isomorphism. In the analytic world, we can use that if the differential of f at any point p vanishes, the preimage of $q = f(p)$ contains p with multiplicity at least 2, a contradiction to the claim that the degree of f is 1. Hence the differential vanishes nowhere, so by the inverse function theorem, the inverse of f is differentiable everywhere.

b) Choose coordinates $z = Z_1/Z_0$ on U_0 and $w = Z_0/Z_1$ on U_1 such that $z = 1/w$ and such that the point $[1 : 0] \in U_0$ is given by $\{z = 0\}$. By definition, the sections of $\mathcal{O}([1 : 0])$ on U_0 are the meromorphic functions $f(z)$ having at most a simple pole at $[1 : 0]$, i.e. of the form $f(z) = s_0(z)/z$ for $s_0(z)$ holomorphic. On the other hand the sections of $\mathcal{O}([1 : 0])$ on U_1 (which does not contain $[1 : 0]$) are simply all the holomorphic functions $s_1(w)$. Thus s_0 and s_1 define compatible sections if $s_0(z)/z = s_1(w)$, that is $s_0 = zs_1 = Z_1/Z_0 s_1$. Comparing with the first exercise, these are the transition functions ϕ_{01} of $\mathcal{O}(1)$. We conclude that $c_1(\mathcal{O}(n)) = c_1(\mathcal{O}(n[1 : 0])) = nc_1(\mathcal{O}([1 : 0])) = n$.

c) In the introduction we have seen that $c_1 : \mathrm{Pic}(C) \to \mathbb{Z}$ is an isomorphism and it sends $\mathcal{O}(1)$ to the generator 1 of $\mathbb{Z}$, so $\mathcal{O}(1)$ generates $\mathrm{Pic}(C)$.

   On the other hand $c_1(\mathcal{O}(a_1p_1 + \ldots + a_rp_r)) = a_1 + \ldots + a_r$, so $\mathcal{O}(a_1p_1 + \ldots + a_rp_r) \cong \mathcal{O}(a_1 + \ldots + a_r)$ by the injectivity of c_1.

3. Let C be a smooth curve and $\mathcal{L}$ a line bundle on C.

a) Show that $\mathcal{L}$ is isomorphic to the trivial line bundle $\mathcal{O}$ iff it has a section $s \in H^0(C,\mathcal{L})$ which vanishes nowhere.

b) For a nonzero meromorphic section s of $\mathcal{L}$ (i.e. a section of $\mathcal{L}$ on a set $U = C \setminus \{p_1, \ldots, p_n\}$) make sure you understand what is meant by its divisor $D = \mathrm{div}(s) \in \mathrm{Div}(C)$. For a second line bundle $\mathcal{L}'$ with nonzero meromorphic section s', we have a meromorphic section $s \otimes s'$ of $\mathcal{L} \otimes \mathcal{L}'$. Show that $\mathrm{div}(s \otimes s') = \mathrm{div}(s) + \mathrm{div}(s')$. Show also that a nonzero meromorphic section s with $\mathrm{div}(s) \geq 0$ (in the sense that all coefficients which appear are nonnegative) extends to a global section on all of C.

c) Show that for a divisor $D = \sum_i a_i p_i$ on C the line bundle $\mathcal{O}(D)$ has a meromorphic section s_D with $\mathrm{div}(s_D) = D$. (*Hint*: Under identifying $\mathcal{O}(D)(U)$ with meromorphic functions on U with zeroes and poles restricted by D, the section s_D corresponds to the function 1.)

d) For a nonzero meromorphic section s of $\mathcal{L}$ as in b) with $D = \mathrm{div}(s)$, show that $\mathcal{L} = \mathcal{O}(D)$. (*Hint*: Show that $\mathcal{L} \otimes \mathcal{O}(-D)$ is trivial by giving a global section that vanishes nowhere.)

e) For $c_1(\mathcal{L}) < 0$ show that $H^0(C, \mathcal{L}) = 0$.

Solution

a) One easily checks that the map $C \times \mathbb{C} \to \mathcal{L}, (p, \lambda) \mapsto \lambda s(p)$ is an isomorphism of line bundles, where $C \times \mathbb{C} \to C$ is the trivial line bundle. It is a worthwile exercise to translate this proof in the other descriptions of line bundles from the lecture.

b) Given a cover $U_1, \ldots, U_r$ of C trivializing $\pi : \mathcal{L} \to C$, a meromorphic section s corresponds to meromorphic functions s_i on the sets U_i via the chosen isomorphisms $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}$. Then we can compute the divisor of the function s_i, giving a linear combination of points in U_i. As $s_i = \phi_{ij} s_j$ with ϕ_{ij} having neither zeroes nor poles on $U_i \cap U_j$, one sees that the coefficient c_p of $p \in U_i \cap U_j$ in $\mathrm{div}(s_i)$ is the same as the coefficient in $\mathrm{div}(s_j)$. In other words, it is irrelevant if we compute c_p using the chart U_i or using U_j. Thus we get a well-defined divisor

$$\mathrm{div}(s) = \sum_{p \in C} c_p p.$$

For another line bundle $\mathcal{L}'$ and a cover $U_1, \ldots, U_r$ trivializing both line bundles, the section $s \otimes s'$ is given in the chart U_i by $s_i \cdot s_i'$ and $\mathrm{div}(s_i \cdot s_i') = \mathrm{div}(s_i) + \mathrm{div}(s_i')$. Note that in this last equality the symbol div means the divisor of a meromorphic function! This equation shows the desired equality $\mathrm{div}(s \otimes s') = \mathrm{div}(s) + \mathrm{div}(s')$.

If $\mathrm{div}(s) \geq 0$, this means that all the meromorphic functions s_i on U_i satisfy $\mathrm{div}(s_i) \geq 0$, so they do not have poles anywhere. But a meromorphic function without poles on U_i actually extends uniquely to a holomorphic function on all of U_i. These extensions then define the extension of s to all of C.

c) For $D = \sum_i a_i p_i$ let $P^+ = \{p_i : a_i \geq 0\}$ and $P^- = \{p_i : a_i < 0\}$. Then the function 1 on C is a meromorphic function and on $U = C \setminus P^-$ it satisfies

$$(\mathrm{div}(1) + D)|_U = (0 + \sum_i a_i p_i)|_U = \sum_{i : p_i \notin P^-} a_i p_i \geq 0,$$

so 1 defines a section of $\mathcal{O}(D)$ on U by definition. Here we just used that when restricting to U by definition all the points p_i with $a_i < 0$ are removed, so the remaining part of the divisor D has only nonnegative coefficients. So 1 is a meromorphic section s_D of $\mathcal{O}(D)$ and we claim that it has the desired property $\operatorname{div}(s_D) = D$.

Indeed, let $q \in C$ be any point and $\Delta \subset C$ a small disc around C with local coordinate z such that $q = \{z = 0\}$. If $q = p_i$ for some i, choose the disc Δ small enough such that no other p_j, $j \neq i$ is contained in Δ. Then $\mathcal{O}(D)$ is trivial when restricted to Δ and we can write

$$\mathcal{O}(D)|_\Delta = \begin{cases} \mathcal{O}_\Delta \cdot 1 & \text{, if } q \neq p_i \text{ for all } i, \\ \mathcal{O}_\Delta \cdot z^{-a_i} & \text{, if } q = p_i. \end{cases}$$

Indeed, let for instance $a_i = 1$, then sections of $\mathcal{O}(D)$ around p_i are meromorphic functions with at most a simple pole at p_i, so they are of the form $h(z) \cdot z^{-1}$ for some holomorphic function h.

But now the section s_D corresponds to the meromorphic function 1, and restricted to Δ, we have $1 = 1 \cdot 1$ if q is not one of the p_i, so the section s_D corresponds to the function 1 on Δ. But the divisor of this meromorphic function is trivial, so by the definition above, the coefficient of q in $\operatorname{div}(s_D)$ is zero.

On the other hand, if $q = p_i$ we have $1 = z^{a_i} \cdot z^{-a_i}$, so the section s_D corresponds to the function z^{a_i} on Δ. This has divisor $a_i q = a_i p_i$ (remember q corresponds to $z = 0$), so the coefficient of p_i in $\operatorname{div}(s_D)$ is a_i as desired.

d) First we remark that $\mathcal{L} \otimes \mathcal{O}(-D) = \mathcal{O}$ implies (by tensoring with $\mathcal{O}(D)$ that $\mathcal{L} = \mathcal{L} \otimes \mathcal{O}(-D) \otimes \mathcal{O}(D) = \mathcal{O}(D)$ as desired. Now on the one hand we have a meromorphic section s of $\mathcal{L}$ with $\operatorname{div}(s) = D$. On the other hand by part c) the line bundle $\mathcal{O}(-D)$ has a meromorphic section s_{-D} with $\operatorname{div}(s_{-D}) = -D$. Thus we get a meromorphic section $s \otimes s_{-D}$ of $\mathcal{L} \otimes \mathcal{O}(-D)$ and by part b) we have

$$\operatorname{div}(s \otimes s_{-D}) = \operatorname{div}(s) + \operatorname{div}(s_{-D}) = D + (-D) = 0.$$

As the zero-divisor is ≥ 0, by b) the section $s \otimes s_{-D}$ extends to a global section of $\mathcal{L} \otimes \mathcal{O}(-D)$ on all of C. But its divisor is trivial, so it has no zeroes anywhere. Then by part a) we know that $\mathcal{L} \otimes \mathcal{O}(-D)$ is the trivial line bundle as desired.

e) Any nonzero section $s \in H^0(C, \mathcal{L})$ satisfies that $\operatorname{div}(s) = \sum_i a_i p_i \geq 0$, as it has no poles anywhere. On the other hand $\mathcal{L} = \mathcal{O}(\operatorname{div}(s))$ by part d) and we see

$$c_1(\mathcal{L}) = c_1(\mathcal{O}(\operatorname{div}(s)) = c_1(\mathcal{O}(\sum_i a_i p_i)) = \sum_i a_i \geq 0,$$

a contradiction to the assumption $c_1(\mathcal{L}) < 0$. Thus $\mathcal{L}$ has no nonzero sections s, finishing the proof.

4. Let $\Lambda = \langle v, w \rangle \subset \mathbb{C}$ be a lattice, then the elliptic curve $E = \mathbb{C}/\Lambda$ has $\omega = dz$ as a basis of $H^1(E, \Omega^1)$ and the cycles $a : [0,1] \to E, t \mapsto tv$ and $b : [0,1] \to E, t \mapsto tw$ as a basis of $H_1(E, \mathbb{Z})$. With respect to these choices show that

$\mathrm{Jac}(E) = \mathbb{C}/H_1(E,\mathbb{Z})$ is canonically isomorphic to E and compute the Abel-Jacobi map

$$\mathrm{AJ} : \mathrm{Div}^0(E) \to \mathrm{Jac}(E).$$

Solution Recall that $H_1(E,\mathbb{Z})$ embeds in $\mathbb{C}$ by $a \mapsto \int_a \omega$ and $b \mapsto \int_b \omega$. Using the choices of ω, a, b above, we see

$$a \mapsto \int_a dz = a(1) - a(0) = v, b \mapsto \int_b dz = b(1) - b(0) = w,$$

so the image of $H_1(E,\mathbb{Z}) = \mathbb{Z}a + \mathbb{Z}b$ in $\mathbb{C}$ is $\mathbb{Z}v + \mathbb{Z}w = \Lambda$, so $\mathrm{Jac}(E) = \mathbb{C}/\Lambda = E$ canonically.

Concerning the Abel-Jacobi map, let $D = \sum_i a_i[z_i]$ be a divisor of degree 0, so $\sum_i a_i = 0$. Then by substracting $0 = \sum_i a_i[0]$ we can write it as $D = \sum_i a_i([z_i] - [0])$. To compute $\mathrm{AJ}(D)$ we need to find a union of paths in C with boundary D and we can choose the union over i of a_i copies of the path γ_i given by $t \mapsto tz_i$, which has as boundary $\gamma_i(1) - \gamma_i(0) = [z_i] - [0]$. Then we compute

$$\mathrm{AJ}(D) = \sum_i a_i \int_{\gamma_i} \omega = \sum_i a_i(z_i - 0) = \sum_i a_i z_i \in \mathbb{C}/\Lambda.$$

Thus we obtain $\mathrm{AJ}(D)$ by summing the points in D (according to their coefficients) using the group law of the elliptic curve.

5. In the lecture you saw that morphisms $f : X \to \mathbb{P}^n$ correspond bijectively to the data of a line bundle $\mathcal{L}$ on X together with sections $s_0, \ldots, s_n$ of $\mathcal{L}$ not vanishing simultaneously. For the following maps f give the corresponding line bundle $\mathcal{L}$ and describe the sections s_i. (*Note*: Sometimes it is not easy to describe the sections s_i explicitly, but (except when indicated) you can describe $\mathrm{div}(s_i)$, which determines s_i up to scaling.)

a) $f : \mathbb{P}^1 \to \mathbb{P}^3, [s:t] \mapsto [s^3 : s^2t : st^2 : t^3]$

b) $f : C \to \mathbb{P}^1$ interpreted as a meromorphic function f on C

c) $\wp : \mathbb{C}/\Lambda \to \mathbb{P}^1$ the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right] \text{ for } z \in \mathbb{C} \setminus \Lambda.$$

Note: Giving s_1 here explicitly in terms of the lattice Λ is quite non-trivial and not part of this exercise!

d) $\wp' : \mathbb{C}/\Lambda \to \mathbb{P}^1$

e) $f : \mathbb{C}/\Lambda \to \mathbb{P}^2, z \mapsto [\wp(z) : \wp'(z) : 1]$ for $z \neq 0$ (*Remark*: As we have seen on Sheet 1, f extends to a function on all of $\mathbb{C}/\Lambda$.)

f)* $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, ([s:t],[u:v]) \mapsto [su : sv : tu : tv]$

Solution A common strategy for the computing the line bundle for a map $f : X \to \mathbb{P}^n$ is to compute the inverse image of the loci $\{[Z_0 : \ldots : Z_n] : Z_i = 0\} \subset \mathbb{P}^n$, with multiplicity. If one of these is a nice divisor D, the line bundle on X is given by $\mathcal{O}(D)$.

a) The inverse image of the locus $\{Z_0 = 0\}$ is given by $3[0:1]$, so we have $\mathcal{L} = \mathcal{O}(3[0:1]) = \mathcal{O}(3)$ and indeed, the components of f are homogeneous polynomials s^3, s^2t, st^2, t^3 of degree 3 in the coordinates s, t, which are sections of $\mathcal{O}(3)$ by Exercise 1.

b) Seeing f as a meromorphic function, we can write $\operatorname{div}(f) = \sum_i a_i p_i - \sum_j b_j q_j$ with $a_i, b_j > 0$. Then $\mathcal{L} = \mathcal{O}(\sum_i a_i p_i) = \mathcal{O}(\sum_j b_j q_j)$ and for the sections s_0, s_1 we can take the meromorphic functions $s_0 = 1, s_1 = f$, which are both global sections of $\mathcal{O}(\sum_j b_j q_j)$. This just corresponds to the fact that we see a meromorphic function f as a morphism $C \to \mathbb{P}^1, p \mapsto [1 : f(p)]$.

c) The preimage of infinity, i.e. $[0:1]$ under $\wp$ is $2[0]$, so $\mathcal{L} = \mathcal{O}(2[0])$. We can take for s_0 the section of $\mathcal{L}$ with $\operatorname{div}(s_0) = 2[0]$ which exists by Exercise 3 c). However, to give s_1 we would need to find the zeroes of the Weierstrass function. All we can say easily is that because $c_1(\mathcal{L}) = 2$, it must be two zeroes z_1, z_2 (with multiplicity). If you are interested in this, you can look at the paper by Duke and Imamoglu linked on the course website.

d) The function $\wp'$ has a triple pole at 0, so $\mathcal{L} = \mathcal{O}(3[0])$. Thus s_0 satisfies $\operatorname{div}(s_0) = 3[0]$. We have also found the zeroes of $\wp'$ on a previous exercise sheet: if $\Lambda = \langle v, w \rangle$ we have

$$\operatorname{div}(s_1) = [v/2] + [w/2] + [(v+w)/2].$$

e) Recall that f extends by $f(0) = [0:1:0]$, so when we compute the preimage of the locus $\{Z_1 = 0\}$, the point 0 does not appear in this preimage and we can use the formula for f given above. But then this preimage is just the locus where $\wp'(z) = 0$ so as seen in the previous exercise part, we have $\mathcal{L} = \mathcal{O}(3[0])$ and the section s_1 with $\operatorname{div}(s_1) = [v/2] + [w/2] + [(v+w)/2]$. On the other hand, in the preimage of $\{Z_0 = 0\}$ we have the two zeroes z_1, z_2 of $\wp(z)$ from part c) and also the point 0, with multiplicity 1, so $\operatorname{div}(s_0) = [z_1] + [z_2] + [0]$. Finally, the only preimage of $\{Z_2 = 0\}$ is at 0 and the multiplicity must be 3 as $\deg(f) = \deg(\operatorname{div}(s_0)) = 3$, so $\operatorname{div}(s_2) = 3[0]$.

f)* We have two projections $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the two different factors of $\mathbb{P}^1 \times \mathbb{P}^1$. Then we note that s defines a section of $\mathcal{O}(1)$ on the first $\mathbb{P}^1$ factor (with coordinates s, t), so $\pi_1^*(s)$ is a section of $\pi_1^*(\mathcal{O}(1))$. On the other hand, on $\mathbb{P}^1$ with coordinates u, v we have the section u of $\mathcal{O}(1)$, so $\pi_2^*(u)$ is a section of $\pi_2^*(\mathcal{O}(1))$. Combining these, we have a section $s_0 = \pi_1^*(s) \otimes \pi_2^*(u)$ of $\mathcal{L} = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$. Similarly, we have $s_1 = \pi_1^*(s) \otimes \pi_2^*(v)$ and so on and one checks that these actually define the map f.

The line bundle $\mathcal{L}$ is sometimes denoted by $\mathcal{O}(1,1)$. Similarly, one writes $\mathcal{O}(a,b) = \pi_1^*\mathcal{O}(a) \otimes \pi_2^*\mathcal{O}(b)$.

Due May 9.

Exercises with * are possibly harder and should be considered as optional challenges.