Algebraic Curves (FS 2017)

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Exercise Sheet 5 - Solutions

1. Let W be a $\mathbb{C}$-vector space of finite dimension $m \geq 2$ and let $V \subset W$ be a vector subspace of dimension $n \geq 1$.

i) Convince yourself that for the inclusion $\mathbb{P}(V) \subset \mathbb{P}(W)$ the set of linear projective planes $\mathbb{P}^n \subset \mathbb{P}(W)$ of dimension n containing $\mathbb{P}(V)$ is isomorphic to $\mathbb{P}(W/V)$ in a canonical way. Show that the map sending a point $[w] \in \mathbb{P}(W) \setminus \mathbb{P}(V)$ to the unique linear $\mathbb{P}^n \subset \mathbb{P}(W)$ containing $[w]$ and $\mathbb{P}(V)$ is given by the map $\mathbb{P}(W) \setminus \mathbb{P}(V) \to \mathbb{P}(W/V)$ induced by the projection $W \to W/V$. This map is called the *projection from* $\mathbb{P}(V)$.

ii) Work out the projection from $\mathbb{P}(V) = \{x_0 = x_1 = \ldots = x_{m-1-n} = 0\} \subset \mathbb{P}^{m-1} = \mathbb{P}(W)$ in coordinates.

Solution

i) A linear projective plane of dimension n containing $\mathbb{P}(V)$ corresponds to a linear subspace $V' \subset W$ of dimension $n+1$ containing V. Under the linear projection $W \to W/V$ the space V' maps to a line in W/V through the origin. Conversely, given a linear subspace of dimension 1 in W/V, its preimage in W is of dimension $n+1$ containing V. This gives the canonical bijection, inducing the map $\mathbb{P}(W) \setminus \mathbb{P}(V) \to \mathbb{P}(W/V)$ described above.

ii) For $W = \mathbb{C}^m$ and $V = \{0\}^{m-n} \times \mathbb{C}^n$, the projection $W \to W/V \cong \mathbb{C}^{m-n}$ can be described as the projection on the first $m-n$ coordinates. Thus we have

$$\mathbb{P}(W) \setminus \mathbb{P}(V) \to \mathbb{P}(W/V) \cong \mathbb{P}^{m-1-n}, [x_0 : \ldots : x_{m-1}] \mapsto [x_0 : \ldots : x_{m-1-n}].$$

2. Let C be a smooth projective curve of genus g.

i) Show that a line bundle $\mathcal{L}$ of degree 0 on C has $h^0(C, \mathcal{L}) = 0$ or 1 and $\mathcal{L}$ is equal to the trivial bundle iff it has $h^0(C, \mathcal{L}) = 1$. *Hint:* Sheet 4, Exercise 3 a).

ii) Show that a line bundle $\mathcal{L}$ of degree $2g-2$ on C is equal to the canonical bundle iff it has $h^0(C, \mathcal{L}) \geq g$ (and in this case of course $h^0(C, \mathcal{L}) = g$).

iii) Let $g \geq 1$ and $p \in C$. Consider the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p) \to \mathcal{O}(p)|_p \to 0. \tag{1}$$

Here the map $\mathcal{O} \to \mathcal{O}(p)$ sends a function f to the section $f \cdot s_p$, where $s_p \in H^0(C, \mathcal{O}(p))$ is a section with $\mathrm{div}(s_p) = [p]$ like in Sheet 4, Exercise 3 d). The map $\mathcal{O}(p) \to \mathcal{O}(p)|_p$ is given by evaluating sections at the point p.

a) Recall that you showed in class that the sections $s_0, \dots, s_{g-1}$ generating $H^0(C, \omega_C)$ have no common zero. Conclude that $h^0(C, \omega_C(-p)) = g-1$.
b) Look at the long exact sequence induced by (1) and the dimensions of the terms appearing in it. Show that the map $H^0(C, \mathcal{O}(p)|_p) \to H^1(C, \mathcal{O})$ appearing in the sequence does not vanish. Conclude that $h^0(C, \mathcal{O}(p)) = 1$. (*Note:* We did not have to use that C is not hyperelliptic!)
c) Show that (still for $g \geq 1$) and $p, q \in C$ we have $\mathcal{O}(p) = \mathcal{O}(q)$ iff $p = q$.

Solution

i) We first show that a degree 0 line bundle $\mathcal{L}$ is trivial iff it has at least one section, then we show that it can never have more than one linearly independent section. Clearly the trivial line bundle has a nonzero section. Conversely, if s is a nonzero section of a degree 0 line bundle $\mathcal{L}$ then $\mathrm{div}(s) = \sum_i a_i[p_i]$ with all $a_i \geq 0$ since s has no poles but also $0 = \deg\mathcal{L} = \sum_i a_i$, so all $a_i = 0$. Thus s vanishes nowhere. By Sheet 4, Exercise 3 a) the line bundle $\mathcal{L}$ is thus trivial. Now if $\mathcal{L}$ had two linearly independent sections s_1, s_2, fix a point $p \in C$ and find some nonzero linear combination $s = \lambda s_1 + \mu s_2$ of them vanishing at p (the linear map $\mathbb{C}^2 \to \mathcal{L}_p \cong \mathbb{C}$ to the fibre $\mathcal{L}_p$ of $\mathcal{L}$ at p given by $(\lambda, \mu) \mapsto \lambda s_1(p) + \mu s_2(p)$ has nontrivial kernel). But now s is a holomorphic section of the degree 0 line bundle $\mathcal{L}$ with at least one zero. By the argument before, this gives a contradiction.

ii) We have $\mathcal{L} \cong \omega_C$ iff $\omega_C \otimes \mathcal{L}^\vee$ is the trivial line bundle. By part i) this is the case iff $h^0(C, \omega_C \otimes \mathcal{L}^\vee) \geq 1$. By Serre duality this is the same as $h^1(C, \mathcal{L}) \geq 1$ and by Riemann-Roch, this is equivalent to

$$h^0(C, \mathcal{L}) = \deg\mathcal{L} - g + 1 + h^1(C, \mathcal{L}) \geq 2g - 2 - g + 1 + 1 = g.$$

iii) a) The space $H^0(C, \omega_C(-p))$ is the kernel of the evaluation map $H^0(C, \omega_C) \to (\omega_C)_p \cong \mathbb{C}$ at p. The fact from class shows that this evaluation map is surjective, hence its kernel has codimension 1 in the g-dimensional space $H^0(C, \omega_C)$.

b) The long exact sequence (with dimensions indicated) is as follows

$$0 \to \underbrace{H^0(\mathcal{O})}_{1} \to \underbrace{H^0(\mathcal{O}(p))}_{?} \to \underbrace{H^0(\mathcal{O}(p)|_p)}_{1} \to \underbrace{H^1(\mathcal{O})}_{h^0(\omega_C)=g} \to \underbrace{H^1(\mathcal{O}(p))}_{h^0(\omega_C(-p))=g-1} \to 0$$

where we used Serre duality and part a) for the last two terms.
Now the map $H^1(\mathcal{O}) \to H^1(\mathcal{O}(p))$ has kernel of dimension exactly 1, so the map $H^0(\mathcal{O}(p)|_p) \to H^1(\mathcal{O})$ must be nonzero, hence injective. This implies that $H^0(\mathcal{O}(p)) \to H^0(\mathcal{O}(p)|_p)$ is the zero-map. But then $H^0(\mathcal{O}(p)) \cong H^0(\mathcal{O})$ is one-dimensional, as claimed.

c) If $\mathcal{O}(p) = \mathcal{O}(q)$ with $p \neq q$, then by Exercise 3, Sheet 4 the line bundle $\mathcal{O}(p)$ has sections s_p, s_q with $\mathrm{div}(s_p) = [p]$, $\mathrm{div}(s_q) = [q]$. Clearly s_p, s_q must be linearly independent, so $h^0(C, \mathcal{O}(p)) \geq 2$, a contradiction.

3. Let C be a smooth projective curve of genus g and let $\mathcal{L}$ be a line bundle on C. Let $s_0, \dots, s_r$ be a basis of $H^0(C, \mathcal{L})$. Remember the following equivalences from the lecture: The data $(C, \mathcal{L}, s_0, \dots, s_r)$ gives

- a map $\varphi : C \to \mathbb{P}^r$ iff the morphism $H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_p) \cong \mathbb{C}$ is surjective for all $p \in C$.
- a map $\varphi : C \to \mathbb{P}^r$ separating points iff the morphism $H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_{p+q}) \cong \mathbb{C}^2$ is surjective for all $p, q \in C$ with $p \neq q$.
- an embedding $\varphi : C \to \mathbb{P}^r$ iff the morphism $H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_{p+q})$ is surjective for all $p, q \in C$.

Here a map φ separates points iff φ is injective and it is an embedding iff its differential is nowhere zero, i.e. the map $\varphi_* : T_pC \to T_{\varphi(p)}\mathbb{P}^r$.

i) Give a proof of the first two criteria and use long exact sequences in cohomology to reformulate all three criteria in terms of the quantities $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p))$ and $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q))$.

ii)* Give a proof of the third criterion. *Hint:* The map $\varphi_* : T_pC \to T_{\varphi(p)}\mathbb{P}^r$ is nonzero iff there is a function f around $\varphi(p)$ vanishing to order 1 at $\varphi(p)$ such that $f \circ \varphi$ also vanishes to order exactly 1 at p. Now in the series of inclusions $H^0(C, \mathcal{L}(-2p)) \subset H^0(C, \mathcal{L}(-p)) \subset H^0(C, \mathcal{L})$, where are the sections s of $\mathcal{L}$ vanishing exactly to order 1 at p? You might want to show and use the general fact that for any line bundle $\mathcal{L}'$ we have $h^0(C, \mathcal{L}') - h^0(C, \mathcal{L}'(-p)) = 0$ or 1, i.e. h^0 can change by at most 1 when twisting by a point.

iii) Use Serre duality to show that any line bundle $\mathcal{L}'$ with $\deg\mathcal{L}' > 2g - 2$ satisfies $h^1(C, \mathcal{L}') = 0$. What is $h^0(C, \mathcal{L}')$ in this case?

iv) Conclude that for $\deg\mathcal{L} \geq 2g$ the line bundle $\mathcal{L}$ gives a map to projective space and for $\deg\mathcal{L} \geq 2g + 1$ this map is an embedding.

Solution

i) A linear map to $\mathbb{C}$ is surjective iff it is nonzero, so the first criterion asks that for every point p there should be a section $s \in H^0(C, \mathcal{L})$ with $s(p) \neq 0$. This is equivalent to asking that not all basis elements s_i vanish simultaneously at p and this in turn was equivalent to $(C, \mathcal{L}, s_0, \dots, s_r)$ defining a map.

For the second criterion, assume first that φ separates points and let $p, q \in C$ be distinct. Then $\varphi(p), \varphi(q)$ are also distinct. Let $\mathcal{L}_p, \mathcal{L}_q \cong \mathbb{C}$ be the fibres of $\mathcal{L}$ at p, q, then we need to show that the evaluation map $H^0(C, \mathcal{L}) \to \mathcal{L}_p \oplus \mathcal{L}_q$ is surjective. This is equivalent to finding sections $s_p, s_q \in H^0(C, \mathcal{L})$ such that $s_p(p) \neq 0, s_p(q) = 0$ and $s_q(p) = 0, s_q(q) \neq 0$. But since $\varphi(p), \varphi(q)$ are distinct points in $\mathbb{P}^r$, we can find linear forms $t_p = a_0x_0 + a_1x_1 + \dots + a_rx_r$ and $t_q = b_0x_0 + \dots$ such that t_p vanishes at q but not at p and t_q vanishes at p but not q. Now we can see t_p, t_q as sections of $\mathcal{O}(1)$ and we define $s_p = \varphi^*t_p$, $s_q = \varphi^*t_q$. These sections satisfy the required conditions. This shows the second criterion.

The converse direction works similarly, where we use the fact that the pullback map $\varphi^* : H^0(\mathbb{P}^r, \mathcal{O}(1)) \to H^0(C, \mathcal{L} = \varphi^*(\mathcal{O}(1)))$ is an isomorphism. Indeed, the basis $x_0, \ldots, x_r$ on the left is pulled back exactly to the basis $s_0, \ldots, s_r$ on the right by definition of φ.

Now looking at the long exact sequence in cohomology for

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0$$

we obtain

$$0 \to H^0(C, \mathcal{L}(-p)) \to H^0(C, \mathcal{L}) \to \mathcal{L}_p \cong \mathbb{C} \to \cdots .$$

By linear algebra, the map on the right is surjective iff its kernel, given by $H^0(C, \mathcal{L}(-p))$, has codimension 1 in $H^0(C, \mathcal{L})$, in other words $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = 1$. A similar argument using the exact sequence

$$0 \to \mathcal{L}(-p-q) \to \mathcal{L} \to \mathcal{L}|_{p+q} \to 0$$

shows that the second and third criteria are equivalent to asking $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p-q)) = 2$ for all $p \neq q$ or for all p, q, respectively.

ii)* We first prove the auxilliary fact that $h^0(C, \mathcal{L}') - h^0(C, \mathcal{L}'(-p)) = 0$ or 1 for any line bundle $\mathcal{L}'$. Indeed, looking at the long exact sequence in the solution of i), the codimension of $H^0(C, \mathcal{L}(-p)) \subset H^0(C, \mathcal{L})$ can either be 0 or 1, depending on whether $H^0(C, \mathcal{L}) \to \mathcal{L}_p \cong \mathbb{C}$ is the zero map or surjective.

Now for the criterion, as described in the hint the map $\varphi_* : T_pC \to T_{\varphi(p)}\mathbb{P}^r$ is nonzero iff there is a (linear) function $f = a_0x_0 + \ldots + a_rx_r$ vanishing at $\varphi(p)$ such that $f \circ \varphi$ vanishes to order exactly 1 at p. Now we can interpret f as a section of $\mathcal{O}(1)$ and then $f \circ \varphi$ corresponds to the pullback section $\varphi^* f = a_0s_0 + \ldots + a_rs_r$. The condition that it vanishes to order exactly 1 at p is equivalent to this section being in $H^0(C, \mathcal{L}(-p))$ but not $H^0(C, \mathcal{L}(-2p))$, so $H^0(C, \mathcal{L}(-2p)) \subset H^0(C, \mathcal{L}(-p))$ is a strict inclusion. Moreover by the first criterion we also know that $H^0(C, \mathcal{L}(-p)) \subset H^0(C, \mathcal{L})$ is a strict inclusion. Also, by the auxilliary fact both inclusions can have codimension at most 1, so they have codimension exactly 1. Thus $h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-2p)) = 2$. By the argument from i) this is equivalent to the map $H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_{2p})$ being surjective.

iii) By Serre duality $h^1(C, \mathcal{L}') = h^0(C, \omega_C \otimes (\mathcal{L}')^\vee)$. But

$$\deg \omega_C \otimes (\mathcal{L}')^\vee = 2g - 2 - \deg \mathcal{L}' < 0,$$

so this line bundle cannot have any nonzero sections, hence $h^0(C, \omega_C \otimes (\mathcal{L}')^\vee) = 0$. By Riemann-Roch we have

$$h^0(C, \mathcal{L}') = h^0(C, \mathcal{L}') - h^1(C, \mathcal{L}') = \deg \mathcal{L}' - g + 1.$$

iv) The degree condition ensures that we can apply part iii) to the line bundles $\mathcal{L}, \mathcal{L}(-p)$ in the first case and $\mathcal{L}, \mathcal{L}(-p-q)$ in the second and indeed obtain

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}(-p)) = (\deg \mathcal{L} - g + 1) - (\deg \mathcal{L} - 1 - g + 1) = 1$$

and

$$h^0(C,\mathcal{L}) - h^0(C,\mathcal{L}(-p-q)) = (\deg\mathcal{L} - g + 1) - (\deg\mathcal{L} - 2 - g + 1) = 2.$$

By the reformulation in part i) this gives that $\mathcal{L}$ defines a map to $\mathbb{P}^{\deg\mathcal{L}-g}$ in the first case and that this map is an embedding in the second case.

4. Recall that a curve C is hyperelliptic iff it admits a degree two map to $\mathbb{P}^1$.

i) Show that every curve of genus 0 is hyperelliptic.

ii) Show that every curve of genus 1 is hyperelliptic.

iii) Show that every curve of genus 2 is hyperelliptic.

iv) Let X be a variety, $\mathcal{L}$ a line bundle on X and assume $f : X \to \mathbb{P}^n$ is given by sections $s_0, \dots, s_n$ of $\mathcal{L}$. Show that the following are equivalent:

   a) $f(X) \subset \mathbb{P}^n$ is *nondegenerate*, i.e. there exists no hyperplane in $\mathbb{P}^n$ containing $f(X)$.

   b) The map $H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(X, f^*(\mathcal{O}(1))) = H^0(X,\mathcal{L})$ induced by pullback via f is injective.

   c) The sections $s_0, \dots, s_n$ are linearly independent.

   In these cases, also the map f itself is called nondegenerate.

v) Show that every nondegenerate map $f : \mathbb{P}^1 \to \mathbb{P}^d$ of degree d is - after a linear change of coordinates on $\mathbb{P}^d$ - a rational normal curve, i.e. the map $[s:t] \mapsto [s^d : s^{d-1}t : s^{d-2}t^2 : \dots : st^{d-1} : t^d]$. We note that for $d \geq 1$ this map is an isomorphism onto its image. *Hint:* Show that a linear change of coordinates simply means that you take corresponding linear combinations of the sections of a line bundle defining f.

vi) Prove that for a hyperelliptic curve C of genus $g \geq 2$ with hyperelliptic map $\varphi : C \to \mathbb{P}^1$, the composition $f \circ \varphi : C \to \mathbb{P}^{g-1}$ of φ with a nondegenerate map $f : \mathbb{P}^1 \to \mathbb{P}^{g-1}$ of degree $g-1$ is (isomorphic to) the canonical map, i.e. the map induced by the line bundle ω_C and its g sections. *Hint:* First show that the line bundle $\mathcal{L}$ on C inducing the map $f\circ\varphi$ has degree $2g-2$, then use Exercise 2 ii) above.

vii) Conclude that for $g(C) \geq 2$ the hyperelliptic map $C \to \mathbb{P}^1$ is unique up to change of coordinates on $\mathbb{P}^1$.

Solution

i) A curve C of genus 0 is isomorphic to $\mathbb{P}^1$ and the map $\mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z^2$ is of degree 2.

ii) Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, then the Weierstrass function $\wp$ is a meromorphic function, i.e. a map $\wp : E \to \mathbb{P}^1$. Since the preimage of ∞ is $2[0]$, the map $\wp$ is of degree 2. Note that $\wp$ is exactly invariant under the involution $z \mapsto -z$ on E, which is the *hyperelliptic involution.*

iii) The line bundle ω_C of a genus 2 curve C has degree $2\cdot 2 - 2 = 2$ and $g = 2$ sections, not vanishing simultaneously, so the canonical map $C \to \mathbb{P}^1$ has degree 2.

iv) We show the equivalence of the negations of the above statements: $f(X) \subset \mathbb{P}^n$ is degenerate iff there is a hyperplane containing it. Hyperplanes are cut out by sections s of $\mathcal{O}(1)$, so this is equivalent to the existence of a section s which pulls back to zero under f, i.e. a nonzero element of the kernel of $H^0(\mathbb{P}^n, \mathcal{O}(1)) \to H^0(X, \mathcal{L})$. But this map is simply given by sending the basis $x_0, \ldots, x_n$ on the left to the sections $s_0, \ldots, s_n$. Thus this map has nontrivial kernel iff $s_0, \ldots, s_n$ are linearly dependent.

v) A nondegenerate map $f : \mathbb{P}^1 \to \mathbb{P}^d$ of degree d is, by definition, given by a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ of degree d together with $d+1$ sections $s_0, \ldots, s_d$ of $\mathcal{L}$. Now as we have seen, every line bundle on $\mathbb{P}^1$ is of the form $\mathcal{O}(m)$, so necessarily $\mathcal{L} \cong \mathcal{O}(d)$. By part iv) the sections $s_0, \ldots, s_d$ are linearly independent sections in $H^0(\mathbb{P}^1, \mathcal{O}(d))$, but this space has dimension $d+1$, so they form a basis. Hence, we can express the usual basis elements $s^d, s^{d-1}t, \ldots, st^{d-1}, t^d$ of $H^0(\mathbb{P}^1, \mathcal{O}(d))$ as linear combinations of $s_0, \ldots, s_d$. Then the invertible $(d+1) \times (d+1)$ matrix defining these linear combinations induces a linear change of coordinates $\psi : \mathbb{P}^d \to \mathbb{P}^d$ and $\psi \circ f$ then has exactly the form $[s : t] \mapsto [s^d : s^{d-1}t : s^{d-2}t^2 : \ldots : st^{d-1} : t^d]$.

vi) The map $f \circ \varphi$ is given by a line bundle $\mathcal{L} = (f \circ \varphi)^* \mathcal{O}(1)$ with sections $s_0, \ldots, s_{g-1}$ satisfying $s_i = (f \circ \varphi)^* x_i$. We need to show that $\mathcal{L} \cong \omega_C$ and that the s_i are a basis of $H^0(C, \omega_C)$.

Since f is degree $g-1$, it satisfies $f^*\mathcal{O}(1) \cong \mathcal{O}(g-1)$. Assume that the hyperelliptic map φ is given by a line bundle $\mathcal{M}$ (of degree 2), then we have

$$\begin{aligned}\mathcal{L} &= (f \circ \varphi)^* \mathcal{O}(1) = \varphi^* f^* \mathcal{O}(1) = \varphi^* \mathcal{O}(g-1) = \varphi^* \mathcal{O}(1)^{\otimes g-1} \\ &= (\varphi^* \mathcal{O}(1))^{\otimes g-1} = \mathcal{M}^{\otimes g-1}.\end{aligned}$$

Since $\mathcal{M}$ has degree 2, the line bundle $\mathcal{L}$ has degree $2g-2$.

By Exercise 2 ii) we have $\mathcal{L} \cong \omega_C$ if we can show $h^0(C, \mathcal{L}) \geq g$. But note that we can obtain sections of $\mathcal{L}$ by pullback via $f \circ \varphi$ and by part iv) we have that the map $H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) \to H^0(C, \mathcal{L})$ is injective (note that f is nondegenerate and φ is surjective, so $f \circ \varphi(C)$ is nondegenerate). Since $\dim H^0(\mathbb{P}^{g-1}, \mathcal{O}(1)) = g$, we have showed that indeed $h^0(C, \mathcal{L}) \geq g$. Moreover, again by part iv), the sections $s_0, \ldots, s_{g-1}$ are linearly independent, hence a basis of $H^0(C, \omega_C)$. Thus $f \circ \varphi$ is isomorphic to the canonical map of C.

vii) We have just seen that for a hyperelliptic curve C of genus $g \geq 2$, its canonical map $\Psi : C \to \mathbb{P}^{g-1}$ factors through any hyperelliptic map $C \to \mathbb{P}^1$ and an embedding $\mathbb{P}^1 \to \mathbb{P}^{g-1}$. But this means that the image of Ψ is (abstractly) isomorphic to $\mathbb{P}^1$ and hence any hyperelliptic map is isomorphic to $\Psi : C \to \Psi(C)$. But this Ψ is canonical (in both senses of the word) and does not depend on any choices, so the hyperelliptic map is unique up to isomorphisms of $\mathbb{P}^1$.

Due May 9.

Exercises with * are possibly harder and should be considered as optional challenges.