Algebraic Curves (FS 2017)

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Exam preparation sheet

In all of the following questions, "curve" means "irreducible nonsingular projective curve".

1. Show that any curve of genus 0 is isomorphic to $\mathbb{P}^1$.
2. Show that a line bundle $\mathcal{L}$ of degree 0 on a curve C has $h^0(C, \mathcal{L}) = 0$ or 1 and $\mathcal{L}$ is the trivial line bundle iff $h^0(C, \mathcal{L}) = 1$.
3. Let $\Lambda = \mathbb{Z}v + \mathbb{Z}w \subset \mathbb{C}$ be a lattice, with $v, w \in \mathbb{C}$. Describe the set of morphisms from $\mathbb{P}^1$ to $\mathbb{C}/\Lambda$.
4. Write down a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree at least 2 and give its divisor $\mathrm{div}(f)$. Also compute the line bundle $f^*\mathcal{O}(1)$.
5. Show that every curve of genus at most 2 is hyperelliptic. *Bonus:* Why is there a curve of genus 3 that is non-hyperelliptic.
6. For an elliptic curve $E = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}v + \mathbb{Z}w \subset \mathbb{C}$ a lattice, go through the definition of the Abel-Jacobi map and show that $\mathrm{Jac}^0(E) \cong E$.
7. Let E be an elliptic curve and $p \in E$ a point. Compute $h^0(E, \mathcal{O}(kp))$ for all $k \in \mathbb{Z}$.
8. Show, using only the Riemann-Roch theorem and Serre duality, that every line bundle $\mathcal{L}$ on a curve C has a meromorphic section.
9. Show that for a curve C, its Picard group $\mathrm{Pic}(C)$ is countable iff the genus of C is 0.
10. For which numbers n does there exist a degree 2 cover $C \to \mathbb{P}^1$ ramified over exactly n different points?
11. Prove that the group of automorphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ is isomorphic to $\mathrm{PGL}_2 = \mathrm{GL}_2/\mathbb{C}^*$. *Hint:* Show that for an isomorphism f with inverse g, one has $f^*\mathcal{O}(1) = \mathcal{O}(1)$.
12. Let $C \subset \mathbb{P}^3$ be the intersection of two nonsingular quadrics $Q_1, Q_2 \subset \mathbb{P}^3$ such that C is a nonsingular, irreducible, projective curve. Compute the genus g of C.
13. Let C be a curve of genus 1. How many line bundles $\mathcal{L} \in \mathrm{Pic}(C)$ satisfy $\mathcal{L}^{\otimes 2} \cong \mathcal{O}$? *Bonus:* What is the answer for a genus g curve C?
14. *Bonus:* Give an argument for the following claim: "The space of hyperelliptic curves of genus $g \geq 2$ has dimension $2g - 1$". You don't have to give precise definitions here.