# Lecture Notes on
# Stochastic Optimal Control
# DO NOT CIRCULATE: Preliminary Version

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# Contents

# Chapter 1

# Deterministic Optimal Control

## 1.1 Setup and Notation

In an optimal control problem, the controller would like to optimize a cost criterion or a *pay-off functional* by an appropriate choice of the *control process*. Usually, controls influence the system dynamics via a set of ordinary differential equations. Then the goal is to minimize a cost function, which depends on the controlled state process.

For the rest of this chapter, all processes defined are Borel-measurable and the integrals are with respect to the Lebesgue measure.

Let $T > 0$ be a fixed finite time horizon, the case $T = \infty$ will be considered later. System dynamics is given by a nonlinear function

$$f : [0, T] \times \mathbf{R}^d \times \mathbf{R}^N \to \mathbf{R}^d.$$

A control process $\alpha$ is any measurable process with values in a closed subset $A \subset \mathbf{R}^N$, i.e., $\alpha : [0, T] \to A$. Then the equation,

$$\begin{aligned} \dot{X}(s) &= f(s, X(s), \alpha(s)) \quad s \in (t, T], \\ X(t) &= x, \end{aligned} \tag{1.1}$$

governs the dynamics of the (controlled) *state process* $X$. To ensure the existence and the uniqueness of the ODE (1.1), we assume uniform Lipschitz (1.2) and linear growth conditions (1.3), i.e for any $s \in (t, T]$ and $\alpha \in A \subset \mathbf{R}^N$,

$$|f(s, x, a) - f(s, y, a)| \leq K_f |x - y|, \tag{1.2}$$

$$|f(s, x, a)| \leq K_f (1 + |x|). \tag{1.3}$$

The unique solution of (1.1) is denoted by

$$X(s; t, x, \alpha) = X^{\alpha}_{t,x}(s), \quad s \in [t, T].$$

Then, the objective is to minimize the cost functional $J$ with respect to the control $\alpha$:

$$J(t,x,\alpha) = \int_t^T L(s, X_{t,x}^{\alpha}(s), \alpha(s))ds + g\left(X_{t,x}^{\alpha}(T)\right), \tag{1.4}$$

where we assume that there exists constants $K_L$ and $K_g$ such that

$$L(s,x,a) \geq -K_L, \qquad g(x) \geq -K_g, \quad \forall s \in [t,T], x \in \mathbf{R}^d, a \in A.$$

The value function is then defined to be the minimum value,

$$v(t,x) = \inf_{\alpha \in \mathcal{A}_{(t,x)}} J(t,x,\alpha), \tag{1.5}$$

where $\mathcal{A}_{(t,x)}$ is the class of admissible controls, which is generally a subset of $L^{\infty}([0,T];A)$. For now, we assume

$$\mathcal{A}_{(t,x)} = \mathcal{A} = L^{\infty}([0,T];A). \tag{1.6}$$

## 1.2 Dynamic Programming Principle

A standard technique in optimal control is analogous to the Euler-Lagrange approach for classical mechanics. Basically, it amounts to finding the zeroes of the functional derivative of the above problem with respect to the control process and it is called the *Pontryagin maximum principle*. It can be rather powerful especially for deterministic control problems with a convex structure. However, it is not straightforward to generalize to the stochastic control problems. In these notes, we develop a technique called dynamic programming which is analogous to the Jacobi approach.

In the dynamic programming approach one characterizes the minimim value as a function of $(t,x)$ and uses the evolution of $v(t,x)$ in time and space to calculate its value. The key observation is the following fix-point property.

**Theorem 1.** *Dynamic Programming Principle(DPP)*

*For $(t,x) \in [0,T) \times \mathbf{R}^d$, and any $h > 0$ with $t + h < T$, we have*

$$v(t,x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_t^{t+h} L(s, X_{t,x}^{\alpha}, \alpha(s))ds + v(t+h, X_{t,x}^{\alpha}(t+h)) \right\}. \tag{1.7}$$

**Proof.** We will prove the DPP in two steps.

Fix $(t,x), \alpha \in \mathcal{A}$ and $h > 0$. Using the additivity of the integral, we express the cost functional as

$$\begin{aligned} J(t,x,\alpha) &= \int_t^T L(s, X_{t,x}^{\alpha}(s), \alpha(s))ds + g\left(X_{t,x}^{\alpha}(T)\right) \\ &= \int_t^{t+h} L(s, X_{t,x}^{\alpha}(s), \alpha(s))ds \\ &+ \int_{t+h}^T L(s, X_{t,x}^{\alpha}(s), \alpha(s))ds + g\left(X_{t+h, X_{t,x}^{\alpha}(t+h)}^{\alpha}(T)\right). \end{aligned}$$

Notice that the solutions of the ODE (1.1) have the following property

$$X^{\alpha}_{t,x}(s) = X^{\alpha}_{t+h, X^{\alpha}_{t,x}(t+h)}(s) \quad \forall s > t+h, \tag{1.8}$$

by the uniqueness. Hence,

$$\int_{t+h}^{T} L(s, X^{\alpha}_{t,x}(s), \alpha(s))ds + g\left(X^{\alpha}_{t+h, X^{\alpha}_{t,x}(t+h)}(T)\right) = J(t+h, X^{\alpha}_{t,x}(t+h), \alpha).$$

Since $J(t+h, X^{\alpha}_{t,x}(t+h), \alpha) \geq v(t+h, X^{\alpha}_{t,x}(t+h))$ and $\alpha$ is arbitrary, we conclude that

$$v(t,x) \geq \inf_{\alpha \in \mathcal{A}} \left\{ \int_{t}^{t+h} L(s, X^{\alpha}_{t,x}(s), \alpha(s))ds + v(t+h, X^{\alpha}_{t,x}(t+h)) \right\}. \tag{1.9}$$

It remains to prove the equality. The idea is to choose $\alpha$ appropriately so that both sides are equal up to $\epsilon$. We set

$$I(t,x) = \inf_{\alpha \in \mathcal{A}} \left\{ \int_{t}^{t+h} L(s, X^{\alpha}_{t,x}(s), \alpha(s))ds + v(t+h, X^{\alpha}_{t,x}(t+h)) \right\}.$$

Then for any given $\epsilon > 0$, we can find $\alpha^{\epsilon}$ such that

$$\left\{ \int_{t}^{t+h} L(s, X^{\alpha^{\epsilon}}_{t,x}(s), \alpha^{\epsilon}(s))ds + v(t+h, X^{\alpha^{\epsilon}}_{t,x}(t+h)) \right\} \leq I(t,x) + \epsilon.$$

Choose $\beta^{\epsilon} \in \mathcal{A}$ in such a way such that

$$J(t+h, X^{\alpha^{\epsilon}}_{t,x}(t+h), \beta^{\epsilon}) \leq v(t+h, X^{\alpha^{\epsilon}}_{t,x}(t+h)) + \epsilon.$$

Denote by $\nu^{\epsilon} = \alpha^{\epsilon} \oplus \beta^{\epsilon}$ the concatenation of the controls $\alpha^{\epsilon}$ and $\beta^{\epsilon}$, namely

$$\nu^{\epsilon}(s) = \begin{cases} \alpha^{\epsilon}(s) & s \in [t, t+h] \\ \beta^{\epsilon}(s) & s \in [t+h, T] \end{cases}. \tag{1.10}$$

It is clear that $\nu^{\epsilon} \in \mathcal{A}$. In a similar fashion as before

$$\begin{aligned} J(t,x,\nu^{\epsilon}) &= \int_{t}^{t+h} L(s, X^{\alpha^{\epsilon}}_{t,x}(s), \alpha^{\epsilon}(s))ds \\ &+ \int_{t+h}^{T} L\left(s, X^{\beta^{\epsilon}}_{t+h, X^{\alpha^{\epsilon}}_{t,x}(s)}(s), \beta^{\epsilon}(s)\right) ds + g\left(X^{\beta^{\epsilon}}_{t+h, X^{\alpha^{\epsilon}}_{t,x}(s)}(T)\right) \\ &= \int_{t}^{t+h} L(s, X^{\alpha^{\epsilon}}_{t,x}(s), \alpha^{\epsilon}(s))ds + J\left(t+h, X^{\alpha^{\epsilon}}_{t,x}(t+h), \beta^{\epsilon}\right) \\ &\leq I(t,x) + 2\epsilon, \end{aligned}$$

by our choice of controls $\alpha^{\epsilon}$ and $\beta^{\epsilon}$. Therefore, sending $\epsilon \to 0$ we conclude the DPP. □

*Remark.* The Dynamic Programming Principle in deterministic setting essentially does not need any assumptions. All variations can be treated easily. The main hidden assumption is the fact that the concatenated control $\nu^{\epsilon} := \alpha^{\epsilon} \oplus \beta^{\epsilon}$ belongs to the admissible class $\mathcal{A}$. This can be weakened slightly but an assumption of this type is essential.

The general structure is outlined in [16].

## 1.3 Dynamic Programming Equation

In this section we will proceed formally. For simplicity, we will adopt the notation

$$X^{\alpha}_{t,x}(s) = X(s), \quad \forall s \in [t, T].$$

Assuming the value function is sufficiently smooth we consider the Taylor expansion of the value function at $(t, x)$

$$v(t+h, X(t+h)) = v(t,x) + \left(\frac{\partial}{\partial t} v(t,x)\right) h + \nabla v(t,x) \cdot (X(t+h) - x) + o(h), \quad (1.11)$$

where we use the standard convention that $o(h)$ denotes a function with the property that $o(h)/h$ tends to zero as $h$ approaches to zero. Also

$$\begin{aligned} X(t+h) &= x + \int_t^{t+h} f(s, X(s), \alpha(s)) ds \\ &\approx x + f(t, x, \alpha(t))h + o(h), \end{aligned} \quad (1.12)$$

by another formal approximation of the integral. Similarly,

$$\int_t^{t+h} L(s, X(s), \alpha(s)) ds \approx L(t, x, \alpha(t))h + o(h).$$

Substitute these approximation back into the DPP to arrive at

$$v(t,x) = v(t,x) + \inf_{\alpha \in \mathcal{A}} \left\{ \frac{\partial}{\partial t} v(t,x) + \nabla v(t,x) \cdot f(t,x,\alpha(t)) + L(t,x,\alpha(t)) \right\} h + o(h).$$

Now we cancel the $v(t, x)$ terms, divide by $h$ and then send $h \to 0$ to finish deriving the Dynamic Programming Equation (DPE), i.e.

**Dynamic Programming Equation.**

$$-\frac{\partial}{\partial t} v(t,x) + \sup_{a \in A} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} = 0, \quad \forall t \in [0,T), \quad x \in \mathbf{R}^d. \quad (1.13)$$

To solve this equation boundary conditions need to be specified. Towards this it is clear that the value function satisfies

$$v(T, x) = g(x). \quad (1.14)$$

## 1.4 Verification Theorem

The above derivation of the dynamic programming is formal in many respects. This will be made rigorous in later sections using the theory of viscosity solutions. However, this formal derivation is useful even without a rigorous justification whenever the equation admits a smooth solution. In this section we provide this approach. We start by defining classical solutions to the DPE.

**Definition 1.** A function $u \in C^1([0,T) \times \mathbf{R}^d)$ is a classical sub(super)-solution to (1.13) if

$$-\frac{\partial}{\partial t} v(t,x) + \sup_{a \in A} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} \leq (\geq)\, 0, \ \ \forall t \in [0,T), x \in \mathbf{R}^d. \tag{1.15}$$

**Theorem 2.** ***Verification Theorem Part I:***
*Assume that $u \in C^1([0,T) \times \mathbf{R}^d) \cap C([0,T] \times \mathbf{R}^d)$ is a classical subsolution of the DPE (1.13) together with the terminal data $u(T,x) \leq g(x)$. Then*

$$u(t,x) \leq v(t,x) \quad \forall (t,x) \in [0,T] \times \mathbf{R}^d.$$

*Proof.* Let $\alpha \in \mathcal{A}$ be arbitrary and $(t,x) \in [0,T] \times \mathbf{R}^d$. Set

$$H(s) = u(s, X(s)).$$

Clearly, $H(t) = u(t,x)$ and $H(T) \leq g(X(T))$. Also using the ODE (1.1) we obtain,

$$\dot{H}(s) = u_s(s, X(s)) + \nabla u(s, X(s)) \cdot f(s, X(s), \alpha(s)).$$

We use the subsolution property of $u$ to arrive at

$$\dot{H}(s) \geq -L(s, X(s), \alpha(s)).$$

We now integrate to conclude that

$$H(T) - H(t) = \int_t^T \dot{H}(s) ds \geq -\int_t^T L(s, X(s), \alpha(s)) ds.$$

Hence,

$$u(t,x) \leq g(X(T)) + \int_t^T L(s, X(s), \alpha(s)) ds = J(t,x,\alpha). \tag{1.16}$$

Since the control $\alpha$ is arbitrary, we have $u(t,x) \leq v(t,x)$. □

**Theorem 3.** ***Verification Theorem Part II:***
*Suppose that $u \in C^1([0,T) \times \mathbf{R}^d) \cap C([0,T] \times \mathbf{R}^d)$ is a solution of the PDE (1.13) with the terminal equation (1.14). Further assume that there exists $\alpha^* \in \mathcal{A}$ such that*

$$-\frac{\partial}{\partial s} u(s, X^*(s)) - \nabla u(s, X^*(s)) \cdot f(s, X^*(s), \alpha^*(s)) - L(s, X^*(s), \alpha^*(s)) = 0$$

*for Lebesque almost every $s \in [t,T)$, where $X^* = X_{t,x}^{\alpha^*}$. Then $\alpha^*$ is optimal at $(t,x)$ and $v(t,x) = u(t,x)$.*

*Proof.* In the proof of the first part of the verification theorem we use $\alpha^*$ instead of an arbitrary control. Then the inequality (1.16) becomes an equality, i.e.

$$u(t,x) = g(X^*(T)) + \int_t^T L(s, X^*(s), \alpha^*(s)) ds = J(t,x,\alpha^*) \geq v(t,x).$$

By the first part, we deduce that $v(t,x) = u(t,x)$ and $\alpha^*$ is optimal. □

## 1.5 An example: Linear Quadratic Regulator (LQR)

This is an extremely important example that had numerous engineering applications. We take,

$$A = \mathbf{R}^N, \;\; f(t,x,a) = Fx + Ba, \;\; L(t,x,a) = \frac{1}{2}\left[Mx\cdot x + Qa\cdot a\right], \;\; g(x) = Gx\cdot x,$$

where $F, M, G$ are matrices of dimension $d \times d$, $B$ is a $d \times N$ matrix and $Q$ is $d \times d$. We assume that $F$ and $G$ are non-negative definite and $Q$ is positive definite.

Summarizing, for a control process $\alpha : [t,T] \to \mathbf{R}^N$, the state $X^\alpha_{t,x} : [t,T] \to \mathbf{R}^d$ satisfies the ordinary differential equation

$$\begin{aligned} &\dot{X}(s) = FX(s) + B\alpha(s) \quad s \in [t,T], \\ &X(t) = x. \end{aligned} \tag{1.17}$$

Moreover, the cost functional has the form

$$J(s,x,\alpha) = \int_t^T \frac{1}{2}\left[MX(s)\cdot X(s) + Q\alpha(s)\cdot\alpha(s)\right] ds + \frac{1}{2}GX(T)\cdot X(T), \tag{1.18}$$

where $X(s) = X^\alpha_{t,x}$.

By exploting the structure of the ODE (1.17) and the quadratic form of the cost functional (1.18) we have the scaling property for any $\lambda \in \mathbf{R}$,

$$v(t,\lambda x) = \lambda^2 v(t,x).$$

Here the key observation is that because of uniqueness of (1.17), we obtain $X^{\lambda\alpha}_{t,\lambda x}(\cdot) = \lambda X^\alpha_{t,x}(\cdot)$. If $d = 1$, by setting $\lambda = \frac{1}{x}$ we can express the value function as a product of two functions with respect to $x$ and $t$ respectively, i.e.

$$v(t,x) = x^2 v(t,1).$$

Motivated by this result, in higher dimensions we guess a solution of the form

$$v(t,x) = \frac{1}{2}R(t)x\cdot x \tag{1.19}$$

for some nonnegative definite and symmetric $d \times d$ matrix $R(t)$. By plugging (1.19) into the dynamic programming equation,

$$-\frac{1}{2}\dot{R}(t)x\cdot x + \sup_{a\in\mathbf{R}^N}\left\{-R(t)x\cdot(Fx+Ba) - \frac{1}{2}Mx\cdot x - \frac{1}{2}Qa\cdot a\right\} = 0. \tag{1.20}$$

Now we can characterize the optimal $a^*$ inside the supremum by taking the first derivative with respect to $a$ and setting it equal to zero, i.e.

$$a^* = -Q^{-1}B^T R(t)x. \tag{1.21}$$

Inserting (1.21) into the DPE (1.20) yields

$$0 = -\frac{1}{2}\left[\dot{R}(t) + F^T R(t) + R(t)^T F + M - R(t)^T BQ^{-1}B^T R(t)\right] x\cdot x.$$

Moreover, the boundary condition implies that

$$\frac{1}{2}R(T)x \cdot x = \frac{1}{2}Gx \cdot x.$$

According to these arguments a candidate for the value function would be (1.19), where $R(t)$ satisfies the matrix Riccati equation

$$\dot{R}(t) + A^T R(t) + R(t)^T A + M - R(t)^T BQ^{-1}B^T R(t) = 0 \tag{1.22}$$

with the boundary condition $R(T) = G$. By the theory of ordinary differential equations the Riccati equation has a backward solution in time on some maximal interval $(t_{min}, T]$. We claim that this solution is symmetric. Indeed for any solution $R$, its transpose $R^T$ is also a solutions. Therefore by uniqueness $R = R^T$.

Now we invoke the verification theorem to conclude that (1.19) is in fact the value function, and the optimal control is given by

$$\begin{aligned} \alpha^*(s) &= -Q^{-1}B^T R(s)X^*(s), \quad \text{where} \\ \dot{X}^*(s) &= (F - BQ^{-1}B^T R(s))X^*(s). \end{aligned} \tag{1.23}$$

We also claim that $t_{min}$ is $-\infty$ and thus the solution exists for all time. Indeed, the theory of ordinary differential equations state that the solution is global as long as there is no finite time blow up. So to prove that the solution remains finite, we use the trivial control $\alpha \equiv 0$. This yields

$$\begin{aligned} v(t,x) &= \frac{1}{2}R(t)x \cdot x \leq J(t,x,0) \\ &= \int_t^T \frac{1}{2}MX(s) \cdot X(s)ds + \frac{1}{2}GX(T) \cdot X(T), \end{aligned}$$

where

$$\dot{X}(s) = FX(s) \quad \Rightarrow \quad X(s) = e^{F(s-t)}x.$$

This estimate implies

$$v(t,x) \leq \frac{1}{2}\left[\int_t^T \left(e^{F(s-t)}\right)^T M \left(e^{F(s-t)}\right) ds + \left(e^{F(T-t)}\right)^T G \left(e^{F(T-t)}\right)\right] x \cdot x.$$

Therefore, $R(t)$ is positive definite so that we conclude we can extend the solution globally. For a further reference on LQR problem and Riccati equations see Chapter 9 in [1].

## 1.6 Infinite Horizon Problems

In this section we take $T = \infty$ and for a given constant $\beta > 0$, set

$$\begin{aligned} L(s,x,\alpha) &= e^{-\beta s}L(x,\alpha), \\ f(t,x,\alpha) &= f(x,\alpha), \\ g(x) &= 0. \end{aligned} \tag{1.24}$$

Then given the control $\alpha$ the system dynamics is

$$\dot{X}(s) = f(X(s), \alpha(s)) \text{ for } s \in (t, T], \quad X(t) = x \tag{1.25}$$

and value function is given by

$$v(t, x) = \inf_{\alpha} \int_t^{\infty} e^{-\beta s} L(X(s), \alpha(s)) ds. \tag{1.26}$$

We exploit the time homogenous structure of (1.25) to obtain

$$X_{t,x}^{\alpha(\cdot)}(s) = X_{0,x}^{\alpha(\cdot - t)}(s - t) \quad s \in [t, T]. \tag{1.27}$$

Using this transformation we can factor out the value function as

$$v(t, x) = e^{-\beta t} v(0, x), \tag{1.28}$$

since

$$\begin{aligned} J(t, x, \alpha) &= \int_t^{\infty} e^{-\beta s} L(X(s), \alpha(s)) \\ &= e^{-\beta t} \int_0^{\infty} e^{-\beta s'} L\left(X_{0,x}^{\bar{\alpha}}(s'), \bar{\alpha}(s')\right) ds' \\ &= e^{-\beta t} J(0, x, \bar{\alpha}) \end{aligned}$$

by making the change of variables $s - t = s'$ and letting $\bar{\alpha}(s') = \alpha(s' + t)$.

Then we can derive the following dynamic programming principle in a similar fashion as before.

$$v(x) := v(0, x) = \inf_{\alpha} \left\{ \int_0^h e^{-\beta s} L(X(s), \alpha(s)) ds + e^{-\beta h} v(X(h)) \right\}. \tag{1.29}$$

Also by substituting (1.28) into the DPE yield,

$$\beta v(x) + \sup_{a \in A} \{-\nabla v(x) \cdot f(x, a) - L(x, a)\} = 0 \quad x \in \mathbf{R}^d. \tag{1.30}$$

Observe that (1.30) has no boundary conditions and is defined for all $x \in \mathbf{R}^d$. To ensure uniqueness we have to enforce growth conditions.

### 1.6.1 Verification Theorem for Infinite Horizon Problems

This is very similar to the previous verification results. The main difference is the role of the growth conditions. Here we provide one such result.

**Theorem 4.** ***Verification Theorem Part I:***
*Assume that $u \in C^1(\mathbf{R}^N)$ is a smooth (sub)solution of the DPE (1.30) satisfying polynomial growth*

$$u(x) \leq K(1 + |x|^{\nu})$$

*for some $K$ and $\nu > 0$. Also suppose that $f$ is bounded apart from satisfying the usual assumptions. Then*

$$u(x) \leq J(x, \alpha) \ \forall \alpha \quad \Rightarrow \quad u(x) \leq v(x).$$

**Proof.** Let $\alpha$ be arbitrary and $X(\cdot)$ be the solution of (1.25). Then set

$$H(s) = e^{-\beta s} u(X^{\alpha}_{t,x}(s)).$$

Then similar to our analysis before,

$$u(x) \leq e^{-\beta T} u(X(T)) + \int_0^T e^{-\beta s} L(X(s), \alpha(s)) ds.$$

Now we let $T$ go to infinity. The second term on the right hand side tends to $J(0, x, \alpha)$ by monotone convergence theorem, since $L(X(s), \alpha(s))$ is by assumption bounded from below. It remains to show that

$$\limsup_{T \to \infty} e^{-\beta T} u(X(T)) \leq 0.$$

Boundedness of $f$ implies that there exists $K_f$ such that

$$|X(s)| \leq x + K_f s.$$

Then by polynomial growth

$$\begin{aligned} e^{-\beta T} u(X(T)) &\leq e^{-\beta T} K_f (1 + |X(T)|^{\nu}) \\ &\leq e^{-\beta T} K_f \left(1 + (|x| + K_f T)^{\nu}\right) \to 0 \quad \text{as } T \to \infty, \end{aligned}$$

since $\beta > 0$ by assumption. □

*Remark.* The growth condition in infinite horizon problems replaces the boundary conditions imposed in finite horizon problems. The growth condition is needed to show

$$e^{-\beta T} u(X(T)) \to 0 \quad \text{as } T \to \infty.$$

Moreover, the assumption $\beta > 0$ is crucial.

**Theorem 5.** ***Verification Theorem Part II:***
*Under the conditions of Part I, we assume that $u \in C^1(\mathbf{R}^d)$ is a solution of (1.30) and $\alpha^* \in \mathcal{A}$ satisfies*

$$\beta u(X^*(s)) - \nabla u(X^*(s)) \cdot f(X^*(s), \alpha^*(s)) - L(X^*(s), \alpha^*(s)) = 0 \quad \forall s \geq 0 \text{ a.e.},$$

*where $X^*(s)$ is the solution of $\dot{X}^*(s) = f(X^*(s), \alpha^*(s))$ with $X^*(t) = x$. Moreover, if*

$$e^{-\beta T} u(X^*(T)) \to 0 \quad \text{as } T \to \infty,$$

*then $u(x) = v(x)$ and $\alpha^*$ is optimal.*

Proof is exactly as before.

## 1.7 Variants of deterministic control problems

Here we briefly mention about several possible extensions of the theory.

### 1.7.1 Infinite Horizon Discounted Problems

Consider the usual setup in deterministic control problems given in section (1.1) with the difference of minimizing

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} \left\{ \int_t^T \exp\left(-\int_t^s \beta(u,X(u))du\right) L(s,X(s),\alpha(s))ds \right.$$
$$\left. + \exp\left(-\int_t^T \beta(u,X(u))du\right) g(X(T)) \right\},$$

where $\beta(u,X(u))$ is not a constant, but a measurable function. For the ease of notation, let

$$B(t,s) = \exp\left(-\int_t^s \beta(u,X(u))du\right).$$

Then the corresponding dynamic programming principle is given as

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} \left\{ \int_t^{t+h} B(t,s)L(s,X(s),\alpha(s))ds + B(t,t+h)v(t+h,X(t+h)) \right\}.$$

The proof follows the same lines as before by noting the fact that

$$B(t,s) = B(t,t+h)B(t+h,s) \quad \forall s \geq t+h.$$

Moreover, by subtracting $v(t,x)$ from both sides, dividing by $h$ and letting $h$ tend to 0, we recover the corresponding the dynamic programming equation

$$0 = -v_t(t,x) + \sup_{a\in A} \left\{ -\nabla v(t,x)\cdot f(t,x,a) + \beta(t,x,a)v(t,x) - L(t,x,a) \right\}.$$

We emphasize that now there is a zeroth order term (i.e., the term $\beta(t,x,a)v(t,x)$) in the partial differential equation.

### 1.7.2 Obstacle Problem or Stopping time Problems

Let $\varphi$ be a given measurable function. Consider

$$v(t,x) = \inf_{\alpha\in\mathcal{A},\theta\in[t,T]} \left\{ \int_t^{\theta\wedge T} L(s,X(s),\alpha(s))ds + \varphi(\theta,X(\theta)) \right\},$$

where $t \leq \theta \leq T$ is a stopping time. The corresponding DPP is given by

$$v(t,x) = \inf_{\alpha\in\mathcal{A},\theta\in[t,T]} \left\{ \int_t^{\theta\wedge(t+h)} L(s,X(s),\alpha(s))ds \right.$$
$$\left. + \varphi(\theta,X(\theta))\chi_{\{\theta\leq t+h\}} + v(t+h,X(t+h))\chi_{\{t+h<\theta\}} \right\}.$$

Then the DPE is a variational inequality,

$$0 = \max\left\{ -v_t(t,x) + \sup_a \left\{ -\nabla v(t,x)\cdot f(t,x,a) - L(t,x,a) \right\} \ ; \ v(t,x) - \varphi(t,x) \right\}$$

for all $t < T, x \in \mathbf{R}^d$ satisfying the boundary equation

$$v(T,x) = \varphi(T,x) := g(x).$$

By choosing $\theta = t$, we immediately see that $v(t,x) \leq \varphi(t,x)$.

### 1.7.3 Exit Time Problem

We are given an open set $O \subset \mathbf{R}^d$ and boundary values $g(t,x)$. We consider $\theta$ to be an exit time from $O$. Exit times in deterministic problems can be defined as exit from either from an open or a closed set. So we consider all possible exit times and define the possible set of exit times $\tau(t,x,\alpha)$ as,

$$\theta \in \tau(t,x,\alpha) \quad \Rightarrow \quad \begin{array}{cc} \text{either} & \theta = T \text{ and } X^{\alpha}_{t,x}(u) \in \overline{O}\ \forall u \in [t,T], \\ \text{or} & X^{\alpha}_{t,x}(\theta) \in \partial O \text{ and } X(u) \in \overline{O}\ \forall u \in [t,\theta]. \end{array}$$

Then the problem is to minimize

$$v(t,x) = \inf_{\alpha \in \mathcal{A}, \theta \in \tau(t,x,\alpha)} \left\{ \int_t^{\theta} L(s, X(s), \alpha(s)) ds + g(\theta, X(\theta)) \right\}$$

yielding the dynamic programming equation

$$-v_t(t,x) + \sup_{a \in A} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} = 0 \quad \forall t < T, x \in O$$

together with the boundary condition

$$v(t,x) = g(t,x) \quad \forall t < T, x \in \partial O.$$

This boundary condition however has to be analyzed very carefully as in many situations it only holds in a weak sense. In the theory of viscosity solutions we will make this precise.

Next we illustrate with an example that it is not always possible to attain the boundary data pointwise. Let $O = (0,1) \subset \mathbf{R}$ and the terminal time $T = 1$. For a given control $\alpha$ the system dynamics satisfies the ODE

$$\dot{X}(s) = \alpha(s), \quad X(t) = x.$$

The value function is

$$v(t,x) = \inf_{\alpha, \theta \in \tau(t,x,\alpha)} \left\{ \int_t^{\theta} \frac{1}{2} \alpha(s)^2 ds + \chi_{\{X(\theta)=1\}} + X(1) \chi_{\{\theta=1\}} \right\}$$

because $\theta \in \tau(t,x,\alpha)$ implies that either $X(\theta) = 0$, $X(\theta) = 1$ or $\theta = 1$.

We choose $\alpha = 0$, $\theta = 1$ to conclude that $0 \le v(t,x) \le x \le 1$.

We next claim that that optimal controls are constant, i.e. for any $\alpha, \theta(\alpha) \in \tau(t,x,\alpha)$ there exists a constant control $\alpha^*$ and $\theta(\alpha^*) \in \tau(t,x,\alpha^*)$ such that $J(t,x,\alpha^*) \le J(t,x,\alpha)$. Indeed by the convexity of the domain, we construct $\alpha^*$ by choosing the straight line from $(t,x)$ to $(\theta, X(\theta))$ so that $\theta \in \tau(t,x,\alpha^*)$. Then it suffices to show that

$$\frac{1}{2} \int_t^s \alpha(u)^2 du \ge \frac{1}{2} \int_t^s (\alpha^*)^2 du = \frac{1}{2} (\alpha^*)^2 (t-s),$$

which follows from Jensen's inequality and

$$\int_t^{\theta} \alpha(u) du = X(\theta) - x = \int_t^{\theta} \alpha^* du.$$

With this observation one can characterize the value function. The following is the result of the direct calculation which involves the minimization of a function of one variable, namely the constant control. The value function is given by,

$$v(t,x) = \begin{cases} x - \frac{1}{2}(1-t) & x \geq (1-t), \\ \frac{x^2}{2(1-t)} & x \leq (1-t). \end{cases}$$

The correspoding DPE is

$$-v_t(t,x) + \sup_a \left\{ -v_x(t,x)a - \frac{1}{2}a^2 \right\} = 0 \quad \forall t < 1, 0 < x < 1 \tag{1.31}$$

so that the candidate for the optimal control is $a^* = -v_x(t,x)$ which yields the Eikonal equation

$$-v_t(t,x) + \frac{1}{2}v_x(t,x)^2 = 0.$$

Now observe that

$$(v_t, v_x) = \begin{cases} \left(\frac{1}{2}, 1\right) & x \geq 1-t \\ \left(\frac{x^2}{2(1-t)^2}, \frac{x}{1-t}\right) & x < 1-t. \end{cases}$$

Hence $v(t,x)$ is $C^1$ and its solves the DPE (1.31). However, at $x = 1$ the boundary data is not attained, i.e. $v(t,x) = 1 - \frac{1}{2}(1-t) < 1$.

**Theorem 6.** ***Verification Theorem: Part I***
*Suppose $u \in C^1(\overline{O} \times (0,T)) \cap C(\overline{O} \times [0,T])$ is a subsolution of the DPE (1.31) with boundary conditions*

$$u(t,x) \leq g(t,x) \ \forall t \leq T, x \in \partial\overline{O} \quad \text{and} \quad t = T, x \in \overline{O}.$$

*Assume that all coefficients are continuous. Then $u(t,x) \leq v(t,x)$ for all $t \leq T, x \in \overline{O}$.*

**Proof.** Fix $(t,x), \alpha \in \mathcal{A}, \theta \in \tau(t,x,\alpha)$. Set

$$H(s) = u(s, X^\alpha_{t,x}(s))$$

which we differentiate to obtain

$$\dot{H}(s) = u_t(s, X(s)) + \nabla u(s, X(s)) \cdot f(s, X(s), \alpha(s) \geq -L(s, X(s), \alpha(s)),$$

since $u$ is a subsolution of the DPE. Moreover, this inequality holds at the lateral boundary as well because of continuity. Now we integrate from $t$ to $\theta$,

$$\begin{aligned} H(t) = u(t,x) &\leq u(\theta, X(\theta)) + \int_t^\theta L(s, X(s), \alpha(s))ds \\ &\leq g(\theta, X(\theta)) + \int_t^\theta L(s, X(s), \alpha(s))ds = J(t,x,\alpha,\theta). \end{aligned}$$

**Theorem 7.** ***Verification Theorem: Part II:*** *Let $u \in C^1(\overline{O} \times (0,T)) \cap C(\overline{O} \times [0,T])$. Suppose for a given $(t,x)$ there exists an admissible optimal control $\alpha^* \in \mathcal{A}$ and an exit time $\theta^* \in \tau(t,x,\alpha^*)$ such that*

$$-u_t(s, X^*(s)) - \nabla u(s, X^*(s)) \cdot f(s, X^*(s), \alpha^*(s)) - L(s, X^*(s), \alpha^*(s)) \geq 0 \quad t \leq s \leq \theta$$

*and*

$$u(\theta^*, X^*(\theta^*)) \geq g(\theta^*, X^*\theta^*),$$

*then*

$$u(t,x) \geq J(t,x,\alpha^*,\theta^*).$$

### 1.7.4 State Constraints

In this class of problems we are given an open set $O \in \mathbf{R}^d$ and we are not allowed to leave $\overline{O}$. Therefore a control $\alpha$ is *admissible* if

$$\alpha \in \mathcal{A}_{t,x} \quad \Leftrightarrow \quad X^{\alpha}_{t,x}(s) \in \overline{O} \ \forall s \in [t,T].$$

In general $\mathcal{A}_{t,x}$ could be empty. But if for all $x \in \partial\Gamma$ there exists $a \in A$ such that

$$f(t,x,a) \cdot \eta(t,x) < 0 \quad \Rightarrow \quad \mathcal{A}_{t,x} \neq \emptyset,$$

where $\eta(t,x)$ is the unit outward normal. In this case, DPP holds and it is given by

$$-v_t(t,x) + \underbrace{\sup_{a \in A} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\}}_{H(t,x,\nabla v(t,x))} = 0, \quad \forall t < T, x \in O.$$

The boundary conditions state that at $x \in \partial O$,

$$-v_t(t,x) + \underbrace{\sup_{\{a \in A: f(t,x,a)\cdot\eta(t,x)<0\}} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\}}_{H_\partial(t,x,\nabla v(t,x))} = 0.$$

By continuity on $\partial O$ we now have two equations,

$$-v_t(t,x) + H(t,x,,\nabla v(t,x)) = 0 = -v_t(t,x) + H_\partial(t,x,\nabla v(t,x)),$$

so that

$$H(t,x,\nabla v(t,x)) = H_\partial(t,x,\nabla v(t,x)) \quad \forall x \in \partial O.$$

Interestingly this boundary condition is sufficient to characterize the value function uniquely. This will be proved in the context of viscosity solutions later.

### 1.7.5 Singular Deterministic Control

In singular control problems the state process $X$ may become discontinuous in time. To illustrate this, consider the state process

$$\begin{aligned} dX(s) &= f(s,X(s),\alpha(s))ds + \hat{\alpha}(s)\nu(s)ds \\ X(t) &= x, \end{aligned} \tag{1.32}$$

where the controls are

$$\alpha : [0,T] \to \mathbf{R}^N, \quad \hat{\alpha} : [0,T] \to \mathbf{R}^+, \quad \nu : [0,T] \to \Sigma \subset \mathbf{S}^{d-1}.$$

The value function is given by

$$v(t,x) = \inf_{\eta=(\alpha,\hat{\alpha},\nu)} \left\{ \int_t^T L(s,X(s),\alpha(s)) + c(\nu(s))\hat{\alpha}(s)ds + g(X(T)) \right\}.$$

The following DPP holds:

$$v(t,x) = \inf_{\eta} \left\{ \int_t^{t+h} L(s,X(s),\alpha(s)) + c(\nu(s))\hat{\alpha}(s)ds + v(t+h,X(t+h)) \right\}$$

If $v$ is continuous in time, by choosing the controls $\alpha(s) = \alpha_0$, $\hat{\alpha}(s) = \frac{\lambda}{h}$ and $\nu(s) = \nu_0 \in \Sigma$, we let $h \downarrow 0$ to obtain,

$$v(t,x) \leq v(t, x + \lambda\nu_0) + \lambda c(\nu_0) \quad \forall \nu_0 \in \Sigma, \lambda \geq 0,$$

because

$$X(t+h) = x + \lambda\nu_0 + \int_t^{t+h} f(s, X(s), \alpha(s))ds.$$

In addition, if we assume that $v \in C^1$ in $x$, then

$$\underbrace{\sup_{\nu\in\Sigma} \{-\nabla v(t,x) \cdot \nu - c(\nu)\}}_{H_{sing}(\nabla v(t,x))} \leq 0.$$

One could follow a similar analysis by choosing $\hat{\alpha}(t) = 0$, i.e. by restricting oneself to continuous controls to reach

$$-v_t(t,x) + \sup_{a\in\mathbf{R}^N} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} \leq 0.$$

Furthermore, it is true that

$$H_{sing}(\nabla v(t,x)) < 0 \quad \Rightarrow \quad -v_t(t,x) + \sup_{a\in\mathbf{R}^N} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} = 0$$

so that we have a variational inequality for the DPE

$$\max\Big\{ H_{sing}(\nabla v(t,x)), \tag{1.33}$$
$$- v_t(t,x) + \sup_{a\in\mathbf{R}^N} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} \Big\} = 0.$$

This inequality also tells us either jumps or continuous controls are optimal. Within this formulation, there are no optimal controls and nearly optimal controls have large values. Therefore, we need to enlarge the class of controls that not only include absolutely continuous functions but also controls of bounded variation on every finite time interval $[0,t]$. To this aim, we reformulate the system dynamics and the payoff functional accordingly. We take a non-decreasing RCLL $A : [0,T] \to \mathbf{R}^+$ with $A(0_-) = 0$ to be our control and intuitively we think of it as $\hat{\alpha}(s) = \frac{dA(s)}{ds}$ so that

$$dX(s) = f(s, X(s), \alpha(s))ds + \mu(s)dA(s) \tag{1.34}$$
$$J(t,x,\alpha,\mu,A) = \int_t^T L(s, X(s), \alpha(s))ds + \int_t^T c(\mu(s))dA(s) + g(X(T))$$

# Chapter 2

# Some Problems in Stochastic Optimal Control

We start introducing the stochastic optimal control via an example arising from mathematical finance, the Merton problem.

## 2.1 Example: Infinite Horizon Merton Problem

There are two assets that we can trade in our portfolio, namely one risky asset and one riskless asset. We call them stock and bond respectively and we denote them by $S$ and $B$. They are governed by the equations

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dW_t) \\ dB_t &= rB_t dt, \end{aligned} \tag{2.1}$$

where $r$ is the interest rate and $W$ is the Brownian motion. Henceforth, all processes are adapted to the filtration generated by the filtration generated by the Brownian motion. We also suppose that $\mu - r > 0$. Our wealth process, the value of our portfolio, is expressed as

$$\frac{dX_t}{X_t} = rdt + \pi_t[(\mu - r)dt + \sigma dW_t] - \kappa_t dt, \tag{2.2}$$

where the proportion of wealth invested in stock $\{\pi_t\}_{t\geq 0}$ and the consumption as a fraction of wealth $\{\kappa_t\}_{t\geq 0}$ are the controls. $\{\kappa_t\}_{t\geq 0}$ is admissible if and only if it satisfies $\kappa_t \geq 0$. We want to avoid the case of bankruptcy, i.e we want $X_t \geq 0$ almost surely for $t \geq 0$. For now we make the simplication that our controls are bounded, so the admissible class is

$$\mathcal{A}_x = \{(\pi, \kappa) \in L^\infty((0,\infty) \times \Omega, dt \otimes P) : \forall t \geq 0 \ \kappa_t \geq 0, \quad X_x^{\pi,\kappa}(t) \geq 0 \ a.s\}. \tag{2.3}$$

Given the initial endowmnet $x$ the objective is to maximize the utility from

consumption $c(t) = \kappa_t X(t)$

$$v(t,x) = \sup_{\kappa \geq 0, \pi} E\left[\int_0^\infty \exp(-\beta t) U(c(t)) dt\right], \quad \text{where} \tag{2.4}$$

$$U(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \ln(c) & \gamma = 1 \end{cases}. \tag{2.5}$$

Since we take $(\pi, \kappa)$ belong to $L^\infty((0,\infty) \times \Omega, dt \otimes P)$, the SDE (2.2) has a unique solution, because it satisfies the uniform Lipschitz and linear growth conditions. By a homothety argument similar to the deterministic case we have $X_{\lambda x}^{\pi,\kappa}(\cdot) = \lambda X_x^{\pi,\kappa}(\cdot)$, moreover the expoential form of the utility implies that $J(\lambda x, \pi, \kappa) = \lambda^{1-\gamma} J(x, \pi, \kappa)$. We conclude $v(\lambda x) = \lambda^{1-\gamma} v(x)$. By choosing $\lambda x = 1$ we have

$$v(x) = x^{1-\gamma} v(1).$$

Call $p = 1 - \gamma$ and $v(1) = \frac{a}{p}$ so that

$$v(x) = \frac{a}{p} x^p. \tag{2.6}$$

We state the associated dynamic programming principle without going into details, since we will come back to it later.

$$\beta v(x) + \inf_{\kappa \geq 0, \pi} \left\{ \underbrace{v_x x \left(-r - \pi(\mu - r) + \kappa\right) - \frac{1}{2} x^2 \sigma^2 \pi^2 v_{xx}}_{\text{infinitesimal generator}} - \underbrace{\frac{1}{1-\gamma} (\kappa x)^{1-\gamma}}_{\text{running cost}} \right\} = 0. \tag{2.7}$$

Reexpressing it

$$\beta v(x) - r x v_x + \underbrace{\inf_\pi \left\{ -\frac{1}{2} x^2 \sigma^2 \pi^2 v_{xx} - \pi(\mu - r) x v_x \right\}}_{I_1} + \underbrace{\inf_{\kappa \geq 0} \left\{ k x v_x - \frac{1}{p} (Kx)^p \right\}}_{I_2} = 0.$$

We can solve for these two optimization problems separately to get

$$\pi^* = \frac{\mu - r}{\sigma^2 (1-p)}$$
$$\kappa^* = a^{\frac{1}{p-1}},$$

which yield

$$I_1 = -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2 (1-p)} x^p a$$
$$I_2 = \frac{p-1}{p} x^p a^{\frac{p}{p-1}}.$$

Plugging these values back to the DPE we obtain

$$\left( \frac{\beta}{p} - r - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2 (1-p)} - \frac{1-p}{p} a^{\frac{1}{p-1}} \right) a x^p = 0,$$

so that we can explicitly solve for $a$,

$$a = \left[ \frac{p}{1-p} \left\{ \frac{\beta}{p} - r - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2 (1-p)} \right\} \right]^{p-1}.$$

Since $a > 0$ we see that $\beta$ has to be sufficiently large. For example for $0 < p < 1$, we need

$$\beta > rp + \frac{1}{2} p \frac{(\mu - r)^2}{\sigma^2 (1-p)}.$$

We also make the remark that the candidate for the optimal control $\pi^*$ is a constant depending only on the parameters.

## 2.2 Pure Investments and Growth Rate

In this class of problems the aim is to maximize the long rate growth rate

$$J(x, \pi) = \liminf_{T \to \infty} \frac{1}{T} \log \left( E[X_x^\pi(T)^p] \right),$$

where we take $p \in (0, 1)$ and the wealth process $X_x^\pi(t)$ is driven by the SDE

$$\begin{aligned} dX_x^\pi(t) &= X_x^\pi(t)[rdt + (\mu - r)\pi_t dt + \pi_t \sigma dW_t] \\ X_x^\pi(0) &= x. \end{aligned}$$

$\pi$ is the fraction of wealth invested in the risky asset, $r$ is the interest rate, $\mu$ the mean rate of return with $\mu > r$ and $W_t$ is one-dimensional Brownian motion. Observe that we take the consumption rate equal to be zero. The aim is to maximize the long term growth rate $J(x, \pi)$ over the admissible class $L^\infty([0, \infty), \mathbf{R})$, i.e. controls $\pi$ are bounded and the value function is given by

$$V(x) = \sup_{\pi \in L^\infty} J(x, \pi).$$

To characterize the solution of this framework we investigate an auxiliary problem, where we want to solve

$$v(x, t, T) = \sup_{\pi \in L^\infty} E[X_{t,x}^\pi(T)^p]$$

Fix $t = 0$, regard $v(x, 0, T)$ as a function of maturity $T$ and set $w(x, T) = v(x, 0, T)$. After a time reversal, the dynamic programming equation associated with $w$ is given by

$$\begin{aligned} 0 &= \frac{\partial w}{\partial T} + \inf_\pi \left\{ -rxw_x - \pi(\mu - r)xw_x - \frac{1}{2} x^2 \sigma^2 \pi^2 w_{xx} \right\}, \quad x > 0, T > 0 \\ x^p &= w(x, 0). \end{aligned}$$

As in the the Merton problem, there is a homothety argument, yielding

$$w(x, T) = x^p w(1, T)$$

so that we can set $w(1,T) = a(T)$ with the initial condition $a(0) = 1$. Our next task is to derive an ODE for $w$ by plugging it in the dynamic programming equation. It is given by

$$0 = a'(T) + \inf_{\pi} \left\{ -rpa(T) - \pi(\mu - r)pa(T) - \frac{1}{2}\pi^2\sigma^2 p(p-1)a(T) \right\} \tag{2.8}$$
$$1 = a(0)$$

Standard arguments yield a candidate for optimal control $\pi^*$. In fact it is given by

$$\pi^* = \frac{\mu - r}{\sigma^2(1-p)}.$$

We make the remark that it is independent of time and is equal to the Merton $\pi$. Plugging the optimal control $\pi^*$ in (2.8) we get

$$a'(T) - p\left(r + \frac{1}{2}\frac{(\mu - r)^2}{(1-p)\sigma^2}\right)a(T) = 0.$$

Together with the initial condition we can explicitly solve for $a(t) = \exp(\beta^* t)$, where

$$\beta^* = p\left(r + \frac{1}{2}\frac{(\mu - r)^2}{(1-p)\sigma^2}\right) \tag{2.9}$$

Using the auxiliary problem we can characterize the optimal $\pi^*$ and the optimal value of the maximizing the long term growth rate.

**Theorem 8.** *We have that*

$$V(x) = \sup_{\pi \in L^\infty} \liminf_{T\to\infty} \frac{1}{T}\log\left(E[X_x^\pi(T)^p]\right) = \beta^*$$

*and $\pi^*$ is optimal.*

**Proof.** Let $\pi \in L^\infty([0,\infty), \mathbf{R})$ and $x > 0$ be arbitrary. Set

$$Y^\pi(t) = \exp(-\beta^* t)[X_x^\pi(t)]^p.$$

We will show that $Y^\pi(t)$ is a supermartingale for $\pi$ and a martingale for $\pi^*$ found in the auxiliary problem. By applying Ito's formula to $Y(t)$ we obtain

$$dY^\pi(t) = Y^\pi(t)\left[-\beta^* + pr + p(\mu - r)\pi_t + \frac{1}{2}p(p-1)\sigma^2\pi_t^2\right]dt + \sigma p\pi_t Y^\pi(t)dW(t).$$

From the ordinary differential equation (2.8) we see that the drift term of $Y^\pi(t)$ is less than or equal to 0 for arbitrary $\pi$ and exactly equal to zero for $\pi^*$. Moreover, since $\pi$ is bounded, the stochastic integral with respect to Brownian motion is a martingale. Hence, we can conclude that $Y^\pi(t)$ is a supermartingale for arbitrary $\pi$ and it is a martingale for $\pi^*$. Hence,

$$EY_t^\pi \le Y_0^\pi = x^p \quad \Rightarrow \quad E[(X_x^\pi(t))^p] \le x^p \exp(\beta^* t)$$

so that

$$\liminf_{T\to\infty} \frac{1}{T}\log\left(E[(X_x^\pi(t))^p]\right) \le \beta^*,$$

where the equality holds for $\pi^*$. $\square$

## 2.3 Portfolio Selection with Transaction Costs

We consider a market of two assets, a stock and a money market account, where the agent has to pay transaction costs when transferring money from one to the other. Let $X_t^1$ and $X_t^2$ be the amount of wealth available in bond and stock respectively. Their system dynamics are given by the following stochastic differential equations

$$X_t^1 = x^1 + \int_0^t (rX_u^1 - c_u)du - L_t + (1-m^1)M_t - \sum_{s\le t} \xi^1 \chi_{\{dM_s \neq 0\}} \tag{2.10}$$

$$X_t^2 = x^2 + \int_0^t X_{u-}^2 \frac{dS_u}{S_{u-}} + (1-m^2)L_t - M_t - \sum_{s\le t} \xi^2 \chi_{\{dL_s \neq 0\}}. \tag{2.11}$$

In the above system dynamics $L_t$ represents the amount of transfers in dollars from bond to stock, whereas $M_t$ is the amount of transfers from stock to bond. The asset $X_t^i$ receiving the money transfer pays proportional transaction cost $m^i$ and also a fixed transaction cost $\xi^i$. If $\xi^i = 0$, then the problem is a singular control problem and for $m^i = 0$ it is an impulse control problem. We will take $\xi^i = 0$ in our analysis. Also, $r$ is the interest rate and $c_t$ is the consumption process, a nonnegative integrable function on every compact interval $[0,t]$. Moreover, since $L$ and $M$ are the cumulative money transfers, they are non-negative, RCLL and non-decreasing functions. Therefore,

$$\begin{aligned} X^1(0) &= x^1 - L(0) + (1-m^1)M(0) \\ X^2(0) &= x^2 + (1-m^2)L(0) - M(0). \end{aligned}$$

Next we want to define the solvency region $\mathcal{S}$. If we start from a point in $\mathcal{S}$ and we make a transaction we should move to a position of positive wealth in both assets. To illustrate this idea if we transfer $\Delta L$ amount from bond to stock from a position $(x^1, x^2)$, then we end up with $(x^1 - \Delta L, x^2 + (1-m^2)\Delta L)$. We could transfer as much as $\Delta L = x_1$ by noting that we should also satisfy $x^2 + (1-m^2)x^1 \ge 0$. We could do the similar analysis for $\Delta M$ to obtain the solvency region

$$\mathcal{S} = \left\{(x^1, x^2) : x^1 + (1-m^1)x^2 > 0, x^2 + (1-m^2)x^1 > 0\right\}. \tag{2.12}$$

We define the admissible class of controls $\mathcal{A}(x)$ such that $(c, L, M)$ is admissible for $(x^1, x^2) \in \bar{\mathcal{S}}$ if $(X_t^1, X_t^2) \in \bar{\mathcal{S}}$ for all $t \ge 0$. The objective is to maximize

$$v(x) = \sup_{(c,L,M)\in\mathcal{A}(x)} E\left\{\int_0^\infty e^{-\beta t} U(c(t))dt\right\},$$

where $U$ is a utility function and $\beta$ is chosen such that $v(x) < \infty$.

*Remark.* If $(x^1, x^2) \in \partial\mathcal{S}$, the only admissible policy is to jump immediately to the origin and remain there, in particular, $c = 0$.

*Remark.* We always have $v(x^1, x^2) \le v_{\text{Merton}}(x^1 + x^2)$.

There are three different kinds of actions an agent can make, do not transact immediately, transfer money from stock to bond or from bond to stock. Therefore, intuitively there should be three components in the corresponding dynamic

programming equation, i.e.

$$\min\left\{\beta v(x)+\inf_{c\geq 0}\left\{-(rx^1-c)v_{x_1}(x)-\mu x^2 v_{x_2}(x)-\frac{1}{2}\sigma^2(x^2)^2 v_{x_2x_2}(x)-U(c)\right\},\right.$$
$$\left. v_{x_1}(x)-(1-m^2)v_{x_2}(x), v_{x_2}(x)-(1-m^1)v_{x_1}(x)\right\}=0. \quad (2.13)$$

Exactly the same way as in deterministic singular control we can have

$$v(x)\geq v(x^1-\Delta L, x^2+(1-m^2)\Delta L)$$

and by letting $\Delta L\to 0$, we recover

$$0\geq -v_{x_1}(x)+(1-m^2)v_{x_2}(x)=\nabla v(x)\cdot(-1,(1-m^2)).$$

We now take as the utility function

$$U(c)=\frac{c^p}{p} \quad p\in(0,1).$$

By scaling $v(\lambda x)=\lambda^p v(x)$ we can drop the dimension of the problem by one, where we set

$$\lambda=\frac{1}{x^1+x^2}$$

so that

$$v(x)=(x^1+x^2)^p w(z) \quad \text{where } z=\frac{x^1}{x^1+x^2}. \quad (2.14)$$

Here we use the fact that $\frac{x^1}{x^1+x^2}$ and $\frac{x^2}{x^1+x^2}$ add up to 1 so it is sufficient to specify only one of them. The domain of $w(z)$ is easily calculated from the definition of the solvency region to be

$$1-\frac{1}{m^1}\leq z\leq\frac{1}{m^2}.$$

Next we want to derive the dynamic programming equation from $w$ from (2.13). By direct calculation

$$\begin{aligned}
v_{x_1}(x)&=(x^1+x^2)^{p-1}(pw(z)+w'(z)(1-z))\\
v_{x_2}(x)&=(x^1+x^2)^{p-1}(pw(z)+w'(z)z)\\
v_{x_2x_2}(x)&=(x^1+x^2)^{p-1}(p(p-1)w(z)-2z(p-1)w'(z)+z^2w''(z))
\end{aligned}$$

so that

$$\min\left(a_1w(z)+a_2w'(z)-a_3w''(z)+F\Big(pw(z)+(1-z)w'(z)\Big),\right.$$
$$\left. w(z)+a_4w'(z), w(z)-a_5w'(z)\right)=0 \quad (2.15)$$

where

$$\begin{aligned}
a_1&=\left[\beta-p(\mu-(\mu-r)z+\frac{1}{2}(p-1)\sigma^2(1-z)^2)\right]\\
a_2&=\left[(\mu-r)z(1-z)+\sigma^2(p-1)(1-z)^2z\right]\\
a_3&=\frac{1}{2}\sigma^2z^2(1-z)^2, \quad a_4=\frac{1-zm^2}{pm^2}, \quad a_5=\frac{1-m^1(1-z)}{m^1p}
\end{aligned}$$

and

$$F(\xi) := \inf_{c\geq 0}\{c\xi - U(c)\} = \frac{p-1}{p}\xi^{\frac{p}{p-1}}.$$

It is established in [14] and independently in [4] that the value function is concave and is a classical solution of the dynamic programming equation (2.13) with the boundary condition $v(x) = 0$, if the value function is finite.

Moreover, the solvency region is split into three regions, in particular three convex cones, the continuation region $\mathcal{C}$,

$$\mathcal{C} = \left\{x \in \mathcal{S} : \beta v(x) + \inf_{c\geq 0}\left\{-(rx^1 - c)v_{x_1}(x) - \mu x^2 v_{x_2}(x) - \frac{1}{2}\sigma^2(x^2)^2 v_{x_2x_2}(x) - U(c)\right\} = 0\right\},$$

the sell stock region $\mathcal{SB}$,

$$\mathcal{SB} = \left\{x \in \mathcal{S} : v_{x_1}(x) - (1-m^2)v_{x_2}(x) = 0\right\},$$

and the sell bond region $\mathcal{SS}_t$

$$\mathcal{SS}_t = \left\{x \in \mathcal{S} : v_{x_2}(x) - (1-m^1)v_{x_1}(x) = 0\right\}.$$

Equivalently, one can show that

$$\mathcal{C} = \left\{x \in \mathcal{S} : v_{x_1}(x) - (1-m^2)v_{x_2}(x) > 0, v_{x_2}(x) - (1-m^1)v_{x_1}(x) > 0\right\}.$$

The fact that all these regions are cones follow from the fact that

$$\nabla v(\lambda x) = \lambda^{p-1}\nabla v(x),$$

by considering the statement (2.14).

The following statement shows that $\mathcal{SB}$ convex. Similar argument works for $\mathcal{SS}_t$ so that $\mathcal{C}$ is also convex.

**Proposition 1.** Let $x_0 \in \mathcal{SB}$, then $(x_0^1 + t, x_0^2 - (1-m^2)t) \in \mathcal{SB}$ for all $t \geq 0$.

**Proof.** A point belongs to $\mathcal{SB}$ if and only if $\nabla v(x)\cdot(1, -(1-m^2)) = 0$. Therefore define the function

$$h(t) = \nabla v(x_0 + t(1, -(1-m^2)))\cdot(1, -(1-m^2)).$$

We know that $h(0) = 0$ so that concavity of the value function implies

$$h'(t) = (1, -(1-m^2))^T D^2 v(x_0 + t(1, -(1-m^2)))(1, -(1-m^2)) \leq 0.$$

Therefore, $h(t) \leq 0$ for all $t \geq 0$. However, according to (2.13) we have in fact equality. So $x_0 + t(1, -(1-m^2)) \in \mathcal{SB}$ for all $t \geq 0$. □

**Proposition 2.**

$$\mathcal{SB} \cap \mathcal{SS}_t = \emptyset.$$

**Proof.** If $x_0 \in \mathcal{SS}_t \cap \mathcal{SB}$, then $\nabla v(x_0) = (0,0)$, because

$$\nabla v(x_0)\cdot(1, -(1-m^2)) = 0 = \nabla v(x_0)\cdot(-(1-m^1), 1).$$

By using the formulation (2.14) we get that $w'$ as well as $w$ is equal to 0, however this is not possible, because $w > 0$. □

From the results above it follows that there are $r_1, r_2 \in [1 - \frac{1}{m^1}, \frac{1}{m^2}]$ such that

$$\mathcal{P} = \left\{ (x,y) \in \mathcal{S} : r_1 < \frac{x}{x+y} < r_2 \right\},$$
$$\mathcal{SB} = \left\{ (x,y) \in \mathcal{S} : r_2 < \frac{x}{x+y} \right\},$$
$$\mathcal{SS}_t = \left\{ (x,y) \in \mathcal{S} : \frac{x}{x+y} < r_1 \right\}.$$

Next we explore the optimal investment-transaction policy for this problem. The feedback control $c^*$ is the optimizer of

$$\inf_{c \geq 0} \left\{ c v_{x_1}(x) - \frac{c^p}{p} \right\}$$

given by

$$c^* = v_{x_1}(x)^{\frac{1}{p-1}}.$$

We will investigate the optimal controls for the three different convex cones separately, assuming none of them are empty.

1. If $(x^1, x^2) \in \bar{\mathcal{C}}$. Then by a result of Lions and Sznitman, [10], there are non-decreasing, nonnegative and continuous processes $L^*(t)$ and $M^*(t)$ such that $X^*(t)$ solves the equation (2.10) with $L^*(t), M^*(t)$ and $c^*$ such that $X_0^* = (x^1, x^2) \in \bar{\mathcal{C}}$. The solution $X^*(t) \in \bar{\mathcal{C}}$ for all $t \geq 0$ and for $t \leq \tau^*$

$$M^*(t) = \int_0^t \chi_{\{X^*(u) \in \partial_1 \mathcal{C}\}} dM^*(u)$$
$$L^*(t) = \int_0^t \chi_{\{X^*(u) \in \partial_2 \mathcal{C}\}} dL^*(u),$$

   where

$$\partial_i \mathcal{C} = \left\{ (x^1, x^2) \in \mathcal{S} : \frac{x^1}{x^1 + x^2} = r_i \right\} \quad i = 1, 2$$

   and $\tau^*$ is either $\infty$ or $\tau* < \infty$ but $X^*(\tau^*) = (0,0)$. The process $X^*(t)$ is the reflected diffusion process at $\partial\mathcal{C}$, where the reflection angle at $\partial_1\mathcal{C}$ is given by $(1 - m^1, -1)$ and at $\partial_2\mathcal{C}$ by $(-1, 1 - m^2)$. The result of Lions and Sznitman does not directly apply to this setting, because the region $\mathcal{C}$ has a corner at the origin, so that the process $X^*$ has to be stopped when it reaches the origin.

2. In the case $(x^1, x^2)$ belongs to $\overline{\mathcal{SS}_t}$ or $\overline{\mathcal{SB}}$, we make a jump to the boundary of $\mathcal{C}$ and then move as if we started from the continuation region. In particular, if $(x^1, x^2) \in \overline{\mathcal{SB}}$, then there exists $\Delta L > 0$ such that

$$x_{\Delta L} = (x^1 - \Delta L, x^2 + \Delta L(1 - m^2)) \in \partial_2 \mathcal{C}.$$

   On the other hand, if if $(x^1, x^2) \in \overline{\mathcal{SS}_t}$, then there exists $\Delta M > 0$ such that

$$x_{\Delta M} = (x^1 + (1 - m^1)\Delta M, x^2 - \Delta M) \in \partial_1 \mathcal{C}.$$

## 2.4 More on Stochastic Singular Control

In this section first we relate some partial differential equations to stochastic control problems.

1. **Cauchy problem and Feynman-Kac Formula**:
   The Cauchy problem

$$-u_t(t,x) - \frac{1}{2}\Delta u(t,x) = 0 \quad \forall t < T,\ x \in \mathbf{R}^d,$$
$$u(T,x) = \phi(x) \quad x \in \mathbf{R}^d$$

   is related to the Feynman-Kac formula

$$u(t,x) = E\left(\phi(X_{t,x}(T))\right), \quad \text{where}$$
$$dX_t = dW_t, \quad X(t) = x,$$

   since the generator of the Brownian motion $W$ is the Laplacian.

2. **Dirichlet problem, Exit time and Obstacle Problem**:
   Consider the Dirichlet problem on $Q = [0,T) \times \mathcal{O}$ for an open set $\mathcal{O} \subset \mathbf{R}^d$

$$-u_t(t,x) - \frac{1}{2}\Delta u(t,x) = 0 \quad \forall (t,x) \in Q$$
$$u(t,x) = \Psi(t,x) \quad \forall (t,x) \in \partial_p Q$$

   In the exit time problem, we are given a domain $\mathcal{O}$ and upon exit from the domain we collect a penalty. This problem can be formulated as a boundary value problem. In the obstacle problem, we can choose the stopping time to collect the penalty. Therefore, the boundary becomes part of the problem, i.e. we have a free boundary problem.

3. **Oblique Neumann problem and the Reflected Brownian Motion**:
   In the oblique Neumann problem we have

$$-u_t(t,x) - \frac{1}{2}\Delta u(t,x) = 0 \quad \forall (t,x) \in Q$$
$$u(T,x) = \phi(x) \quad \forall x \in \mathcal{O}$$
$$\nabla u(t,x) \cdot \eta(x) = 0 \quad \forall x \in \partial\mathcal{O},$$

   where $\eta : \partial\mathcal{O} \to \mathbf{S}^{d-1}$ is a nice direction, in the sense that

$$\eta(x) \cdot \vec{n}(x) \geq \epsilon > 0 \tag{2.16}$$

   for all $x \in \partial\mathcal{O}$, where $\vec{n}$ is the unit outward normal. The last condition means that the direction $\eta(x)$ is never tangential to the boundary of $\mathcal{O}$. The probabilistic counterpart of the oblique Neumann problem is the reflected Brownian motion, also known as the Skorokhod problem. It arises in singular stochastic control problems and we saw an example of it in the case of the transaction costs. In the Skorokhod problem we are given an open set $\mathcal{O} \subset \mathbf{R}^d$ with a smooth boundary and a direction $\eta(x)$ satisfying (2.16) together with a Brownian motion $W_t$. Then the problem is to find

a continuous process $X(\cdot)$ and an adapted, continuous and non-decreasing process $A(\cdot)$ satisfying

$$\begin{aligned} X(u) &= x + W(u) - W(t) - \int_t^u \eta(X(s))dA(s) \quad \forall u \geq t \\ X(t) &= x \in \mathcal{O}, \quad X(u) \in \overline{\mathcal{O}} \quad \forall t \geq u \\ A(u) &= \int_t^u \chi_{\{X(s)\in\partial\mathcal{O}\}}(s)dA(s). \end{aligned}$$

This problem was studied in [10].

4. **Singular Control**
   In this problem, the optimal solution is also a reflected Brownian motion. However, as in the obstacle problem, the boundary of reflection is also a part of the solution.

Next we illustrate an example of a stochastic singular control problem. Let $W_u$ be a standard Brownian motion adapted to the filtration generated by it. Let the stochastic differential equation

$$\begin{aligned} dX_u &= dW_u + \nu(u)dA_u, \\ X(t) &= x \end{aligned} \tag{2.17}$$

be controlled by a direction $\nu \in \Sigma \subseteq \mathbf{S}^{d-1}$ and non-negative cadlag non-decreasing process $A(u)$ for $u \geq t$. The objective is

$$v(x) = \inf_{\nu,A} E\left\{\int_0^\infty e^{-t}\left(L(X(t))dt + c(\nu(t))dA(t)\right)\right\}. \tag{2.18}$$

Then the dynamic programming equation becomes

$$\max\left\{v(x) - \frac{1}{2}\Delta v(x) - L(x), H(\nabla v(x))\right\} = 0, \tag{2.19}$$

where

$$H(\nabla v(x)) = \sup_{\nu\in\Sigma}\left\{-\nu\cdot\nabla v(x) - c(\nu)\right\}. \tag{2.20}$$

A specific case of the above problem is that when $\Sigma = \mathbf{S}^{d-1}$ and $c(\nu) = 1$ so that $v(x) \leq v(x+a) + a$ for all $x, a$ which is equivalent to

$$|\nabla v(x)| \leq 1.$$

In this case the dynamic programming equation becomes

$$\max\left\{v(x) - \frac{1}{2}\Delta v(x) - L(x), |\nabla v(x)| - 1\right\} = 0 \quad \forall x \in \mathbf{R}. \tag{2.21}$$

There is a regularity result for this class of stochastic singular control problems established by Shreve and Soner, [15].

**Theorem 9.** *For the singular control problem (2.17), (2.18) with the parameters $d = 2$, $\Sigma = \mathbf{S}^{d-1}$ and $c = 1$, if the running cost $L$ is strictly convex, nonnegative with $L(0) = 0$, then there is a solution $u \in C^2(\mathbf{R}^2)$ of (2.21). Moreover,*

$$\mathcal{C} = \left\{x \in \mathbf{R}^2 : |\nabla v(x)| < 1\right\}$$

*is connected with no holes and has smooth boundary $\partial C \in C^\infty$. Moreover,*

$$\eta(x) = \frac{\nabla v(x)}{|\nabla v(x)|}$$

*satisfies*

$$\eta(x) \cdot \vec{n} \geq 0 \quad \forall x \in \partial C$$

*where $\vec{n}$ is the unit outward normal.*

We have also a verification theorem for the stochastic control problem (2.17), (2.18).

**Theorem 10.** ***Verification Theorem Part I:***
*Assume that $L \geq 0, c \geq 0$. Let $u \in C^2(\mathbf{R}^d)$ be a solution of the dynamic programming equation (2.19). We have the following growth conditions*

$$0 \leq u(x) \leq c(1 + L(x)), \tag{2.22}$$

$$0 \leq L(x) \leq c(1 + |x|^m) \quad \textit{for some } c > 0 \textit{ and } m \in \mathbf{N}. \tag{2.23}$$

*Furthermore, we take the admissibility class $\mathcal{A}$ to be*

$$\mathcal{A} = \left\{ A, \nu : E\left( (X_{t,x}^{A,\nu})^m \right) < \infty, \ \forall t \geq 0, m \in \mathbf{N} \right\}. \tag{2.24}$$

*Then we have $u(x) \leq v(x)$ for all $x \in \mathbf{R}^s$.*

**Proof.** Fix $x, (\nu, A) \in \mathcal{A}$ and let

$$Y(t) = \exp(-\beta t) u(X(t)).$$

By Ito's formula we get that

$$\exp(-\beta t)u(X(t)) = u(x) + \int_0^t \exp(-\beta s) \left( -\beta u(X(s)) + \frac{1}{2}\Delta u(X(s)) \right) ds$$
$$+ \int_0^t \exp(-\beta s) \nabla u(X(s)) \cdot \nu(s) dA_s^c + \sum_{s \leq t} \exp(-\beta s) \left\{ u(X(s_- + \nu_s \Delta A_s)) - u(X(s_-)) \right\}$$
$$+ \int_0^t \exp(-\beta s) \nabla u(X(s)) \cdot dW_s. \tag{2.25}$$

By the dynamic programming equation (2.19) and the fact that for all $\nu \in \Sigma$

$$-\nabla u(X(s)) \cdot \nu(X(s)) \leq c(\nu(X(s))),$$

holds, we get that

$$u(x) + \int_0^t \nabla u(X(s)) \cdot dW_s$$
$$\leq \exp(-\beta t) u(X(t)) + \int_0^t \exp(-\beta s) \left( L(X(s)) ds + c(\nu(X(s))) dA_s \right). \tag{2.26}$$

Since $c \geq 0$ and $L \geq 0$, monotone convergence theorem yields as $t \to \infty$

$$E\left[ \int_0^t \exp(-\beta s) \left( L(X(s)) ds + c(\nu(X(s))) dA_s \right) \right] \to J(x, \nu, A).$$

We may assume without loss of generality that

$$E\left[\int_0^\infty \exp(-\beta t)L(X(t))dt\right] < \infty.$$

This implies that there exists a set of deterministic times $\{t_m\}_{m=1}^\infty$ such that $t_m \uparrow \infty$ and

$$E\left[\exp(-\beta t_m)L(X(t_m))\right] \to 0 \quad \text{as} \quad t_m \uparrow \infty.$$

Morever, there exists a sequence of stopping times $\{\theta_n\}$ such that $\theta_n \uparrow \infty$ and

$$E\left[\int_0^{t\wedge\theta_n} \nabla u(X(s)) \cdot dW_s\right] = 0, \quad \forall t \geq 0,$$

since the stochastic integral $\int_0^t \nabla u(X(s)) \cdot dW_s$ is a local martingale. Taking expectations of both sides in (2.26) yields for all $n, m \in \mathbf{N}$

$$\begin{aligned} u(X(t_m \wedge \theta_m) &\leq E\left[\exp(-\beta(t_m \wedge \theta_n)u(X(t_m \wedge \theta_n))\right] \\ &\quad + E\left[\int_0^{t_m\wedge\theta_n} \exp(-\beta s)(L(X(s))ds + c(\nu(X(s))))dA_s\right]. \end{aligned}$$

Then the first step is to show for fixed $m \in \mathbf{N}$

$$E\left[\exp(-\beta(t_m \wedge \theta_n)u(X(t_m \wedge \theta_n))\right] \to E\left[\exp(-\beta t_m u(X(t_m))\right].$$

This follows, because for any $n$

$$u(X(t_m \wedge \theta_n))^2 \leq C(1 + L(X(t_m \wedge \theta_n)))^2 \leq \bar{c}(1 + |X(t_m \wedge \theta_n)|^{2m})$$

so that by the definition of the admissibility class $\mathcal{A}$ we get

$$\sup_n E\left[\exp(-\beta(t_m \wedge \theta_n))u(X(t_m \wedge \theta_n))\right] < \infty.$$

Together with this fact and $\exp(-\beta(t_m\wedge\theta_n))u(X(t_m\wedge\theta_n)) \to \exp(-\beta t_m)u(X(t_m))$ almost surely implies

$$\begin{aligned} u(X(t_m)) &\leq E\left[\exp(-\beta t_m)u(X(t_m))\right] \\ &\quad + E\left[\int_0^{t_m} \exp(-\beta s)(L(X(s))ds + c(\nu(X(s))))dA_s\right]. \end{aligned}$$

Now we are done by the choice of the sequence $t_m$. □

**Theorem 11.** ***Verification Theorem Part II:***
*Under the conditions of the theorem (10) define*

$$\mathcal{C} = \{x : H(\nabla u(x)) < 0\}.$$

*Assume that $\partial\mathcal{C}$ is smooth and on $\partial\mathcal{C}$*

$$0 = H(\nabla u(x)) = -\nabla u(x) \cdot \eta^*(x) - c(\eta^*(x)) = 0$$

*for some* $\eta^*$ *satisfying*

$$\eta^*(x) \cdot \vec{n} \geq \epsilon > 0 \quad \forall x \in \partial\mathcal{C}.$$

*Then the solution of the Skorokhod problem* $X^*(t), A^*(t)$ *for the inputs* $\eta^*$, $\mathcal{C}$ *and* $x \in \mathcal{C}$ *given by*

$$X^*(t) = x + W(t) + \int_0^t \eta^*(X^*(s))dA_s^* \in \overline{\mathcal{C}} \quad \forall t \geq 0$$
$$A^*(t) = \int_0^t \chi_{\{X^*(s)\in\partial\mathcal{C}\}} dA_s^* \quad \forall t \geq 0$$

*satisfies*

$$u(x) = v(x) = E\left[\int_0^\infty \exp(-\beta s)\left(L(X^*(s))ds + c(\eta^*(X^*(s)))dA_s^*\right)\right]$$

*provided that*

$$E\left[\exp(-\beta t)u(X^*(t))\right] \to 0, \text{ as } t \to \infty.$$

# Chapter 3

# HJB Equation and Viscosity Solutions

## 3.1 Hamilton Jacobi Bellman Equation

As before, we consider a system controlled by a stochastic differential equation given as

$$\begin{aligned} dX_s &= f(s, X(s), \alpha(s))ds + \sigma(s, X(s), \alpha(s))dW_s \quad s \in [t, T] \\ X(t) &= x, \end{aligned}$$

where $\alpha \in L^\infty([0,T]; \mathbf{A})$ is the control process, $A$ is a subset of $\mathbf{R}^N$, $W$ is the standard Brownian motion of dimension $d$, $T \in (0, \infty)$ is time horizon. Let $\tau$ be the exit time from $Q := \mathcal{O} \times (0,T)$ for an open set $\mathcal{O} \subseteq \mathbf{R}^d$. Denote the parabolic boundary $\partial_p Q$ to be the set $[0,T] \times \partial\mathcal{O} \cup \{T\} \times \mathcal{O}$. We want to minimize the cost functional

$$J(t, x, \alpha) = E\left[\int_t^\tau \{L(u, X(u), \alpha(u))du + \Psi(\tau, X(\tau))\}\right]$$

*Remark.* Observe the cost functional depends on the underlying probability space $\nu = (\Sigma, \mathcal{F}, P, \{\mathcal{F}_u\}_{t \le u \le T}, W)$.

The corresponding dynamic programming equation for the value function $v$ is called the Hamilton-Jacobi-Bellman (HJB) equation given by

$$-v_t(t,x) + H(t, x, Dv, D^2 v) = 0 \;\; \forall (t,x) \in Q, \tag{3.1}$$

$$v(t,x) = \Psi(t,x) \;\; \forall (t,x) \in \partial_p Q. \tag{3.2}$$

We can express the map $H : ([0,T] \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{M}^d_{sym}) \to \mathbf{R}$ in the above equation as

$$H(t, x, p, \Gamma) = \sup_{a \in A} \left\{ -f(t,x,a) \cdot p - \frac{1}{2}\mathrm{tr}(\sigma\sigma^T(t,x,a)\Gamma) - L(t,x,a) \right\}. \tag{3.3}$$

We denote $\gamma(t,x,a) = (\sigma\sigma^T)(t,x,a)$ and we note $\mathrm{tr}(\gamma\Gamma) = \sum_{i,j=1}^d \gamma_{ij}\Gamma_{ij}$.

**Proposition 3. Properties of the Hamiltonian**

1. The map $(p, \Gamma) \to H(t, x, p, \Gamma)$ is convex for all $(t, x)$.
2. The function $H$ is degenerate parabolic, i.e. for all non-negative definite $B \in \mathcal{M}^d_{sym}$ and $(t, x, p, \Gamma) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{M}^d_{sym}$ we have

$$H(t, x, p, \Gamma + B) \leq H(t, x, p, \Gamma). \tag{3.4}$$

**Proof.** 1. The functions

$$(p, \Gamma) \to -f(t, x, a) \cdot p - \frac{1}{2}\mathrm{tr}(\gamma(t, x, a)\Gamma)$$

are linear for all $(t, x, a)$, therefore convex for all $(t, x, a)$. So taking the supremum over $a \in A$, we conclude that $H$ is convex in $(p, \Gamma)$ for all $(t, x)$.

2. If we can show that $\mathrm{tr}(\gamma(t, x, a)B) \geq 0$ for all $a \in A$, it follows easily that $H$ is degenerate parabolic by the definition of $H$. Since $B$ is symmetric we can express $B = U^* \Lambda U$, where

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d), \quad U = \begin{pmatrix} u^1 \\ u^2 \\ \vdots \\ u^d \end{pmatrix},$$

and $u^i$ are the eigenvectors corresponding to eigenvalues $\lambda_i$. Moreover, $\lambda_i \geq 0$ because $B$ is non-negative definite. Then

$$B = \sum_{i=1}^{d} \lambda_i u^i \otimes u^i \Rightarrow \gamma B = \sum_{i=1}^{d} \lambda_i (\gamma u^i) \otimes u^i.$$

Therefore, by non-negative definiteness of $\gamma$ it follows that

$$\mathrm{tr}(\gamma B) = \sum_{i=1} \lambda_i (\gamma u^i) \cdot u^i \geq 0.$$

□

*Remark.* If $\gamma$ is uniformly elliptic, i.e. there exists $\epsilon > 0$ such that $\gamma(t, x, a) \geq \epsilon I_d$ for all $(t, x) \in \overline{Q}$ and $a \in A$, then

$$H(t, x, p, \Gamma + B) \leq H(t, x, p, \Gamma) - \epsilon \mathrm{tr}(B).$$

Next we state an existence result for the Hamilton-Jacobi-Bellman equation (3.1),(3.2) for uniformly parabolic $\gamma$ due to Krylov, [9].

**Theorem 12.** *Assume that*

1. *$\gamma$ is uniformly elliptic for $(t, x, a) \in Q \times A$.*
2. *$A$ is compact.*
3. *$\mathcal{O}$ is bounded with $\partial\mathcal{O}$ a manifold of class $C^3$.*

4. $f, \gamma, L \in C^{1,2}$ *on* $\overline{Q} \times A$.

5. $\psi \in C^3(\overline{Q})$.

*Then the Hamilton-Jacobi-Bellman equation (3.1),(3.2) has a unique solution* $v \in C^{1,2}(Q) \cap C(\overline{Q})$.

If $A$ is compact, then we can choose the feedback control

$$\alpha^*(t,x) \in \operatorname{argmax} \left\{ \alpha \in A : -f(t,x,a) \cdot \nabla v(t,x) - \frac{1}{2}\operatorname{tr}\left(\gamma(t,x,a) D^2 v(t,x)\right) - L(t,x,a) \right\}$$

to be Borel measurable. Such a function $\alpha^* : \overline{Q} \to A$ is called a Markov control.

## 3.2 Viscosity Solutions for Deterministic Control Problems

In this section we will define and explore some properties of viscosity solutions for deterministic control problems.

### 3.2.1 Dynamic Programming Equation

We return back to deterministic setup where the system dynamics $X^{\alpha}_{t,x}$ is governed by

$$\begin{aligned} \dot{X}(s) &= f(s, X(s), \alpha(s)) \quad s \in (t,T], \\ X(t) &= x, \end{aligned} \tag{3.5}$$

and the control process $\alpha$ takes values in a compact interval $A \subset \mathbf{R}^N$. Moreover, we assume that $\alpha \in L^\infty([0,T], A)$. The cost functional is given by

$$v(t,x) = \inf_{\alpha \in L^\infty([0,T],A)} \left\{ \int_t^T L(u, X(u), \alpha(u)) du + g(X(T)) \right\}$$

Also, suppose that $f$ and $L$ satisfies the uniform Lipschitz condition (1.2) and $f$ is growing at most linearly (1.3). For this problem, dynamic programming principle was proved in Section (1.2). However, we used formal arguments to derive the associated dynamic programming equation

$$-\frac{\partial}{\partial t} v(t,x) + \sup_{a \in A} \{-\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a)\} = 0, \quad \forall t \in [0,T), \quad x \in \mathbf{R}^d. \tag{3.6}$$

Now we use viscosity solutions to complete the argument. The main idea is to replace the value function with a test function, since we do not have a priori any information about the regularity of the value function. First we develop the definition of viscosity solution for a continuous function.

**Definition 2. Viscosity sub-solution**:
We call $v \in C(\overline{Q_0})$ a viscosity sub-solution of (3.6), if for $(t_0, x_0) \in Q_0 = [0,T) \times \mathbf{R}^d$ and $\varphi \in C^\infty(\overline{Q_0})$ satisfying

$$0 = (v - \varphi)(t_0, x_0) = \max\left\{(v - \varphi)(t,x) : (t,x) \in \overline{Q_0}\right\}$$

we have that

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0)+\sup_{a\in A}\{-\nabla\varphi(t_0,x_0)\cdot f(t_0,x_0,a)-L(t_0,x_0,a)\}\leq 0.$$

Analogously, we define the viscosity supersolution,

**Definition 3. Viscosity super-solution**:
We call $v\in C(\overline{Q_0})$ a viscosity super-solution of (3.6), if for $(t_0,x_0)\in Q_0=[0,T)\times\mathbf{R}^d$ and $\varphi\in C^\infty(\overline{Q_0})$ satisfying

$$0=(v-\varphi)(t_0,x_0)=\min\left\{(v-\varphi)(t,x):(t,x)\in\overline{Q_0}\right\}$$

we have that

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0)+\sup_{a\in A}\{-\nabla\varphi(t_0,x_0)\cdot f(t_0,x_0,a)-L(t_0,x_0,a)\}\geq 0.$$

We call $v$ a viscosity solution if it is both a viscosity sub-solution and a viscosity super-solution.

**Theorem 13.** ***Dynamic Programming Equation***:
*Assume that $v\in C(\overline{Q_0})$ satisfies the dynamic programming principle*

$$v(t,x)=\inf_{\alpha\in L^\infty([0,T];A)}\left\{\int_t^{t+h}L(s,X(s),\alpha(s))ds+v(t_0+h,X(t_0+h))\right\} \tag{3.7}$$

*then $v$ is a viscosity solution of the dynamic programming equation (3.6).*

**Proof.** First we will prove that $v$ is a viscosity subsolution. So suppose $(t_0,x_0)\in Q_0$ and $\varphi\in C^\infty(\overline{Q_0})$ satisfy

$$0=(v-\varphi)(t_0,x_0)=\max\left\{(v-\varphi)(t,x):(t,x)\in\overline{Q_0}\right\}. \tag{3.8}$$

Then our goal is to show that

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0)+\sup_{a\in A}\{-\nabla\varphi(t_0,x_0)\cdot f(t_0,x_0,a)-L(t_0,x_0,a)\}\leq 0.$$

We know that (3.8) implies $v(t_0,x_0)=\varphi(t_0,x_0)$ and $v(t,x)\leq\varphi(t,x)$. Putting these observations into the dynamic programming principle (3.7) and choosing a constant control $\alpha=a$, we obtain

$$\varphi(t_0,x_0)\leq\int_{t_0}^{t_0+h}L(s,X(s),a)ds+\varphi(t_0+h,X(t_0+h)). \tag{3.9}$$

We have the Taylor expansion of $\varphi(t_0+h,X(t_0+h))$ around $(t_0,x_0)$ in integral form

$$\varphi(t_0+h,X(t_0+h))=\varphi(t_0,x_0)+\int_{t_0}^{t_0+h}\left\{\frac{\partial}{\partial t}\varphi(u,X^a(u))+\nabla\varphi(u,X^a(u))\cdot f(u,X^a(u),a)du\right\}.$$

Substitute the above expression in (3.9), cancel the $\varphi(t_0,x_0)$ terms and divide by $h$. The result is

$$0\leq\frac{1}{h}\int_{t_0}^{t_0+h}\left\{L(u,X^a(u),a)+\frac{\partial}{\partial t}\varphi(u,X^a(u))+\nabla\varphi(u,X^a(u))\cdot f(u,X^a(u),a)\right\}du.$$

Observe that $L$, $f$, $\frac{\partial}{\partial t}\varphi$, $\nabla\varphi$ are jointly continuous in $(t,x)$ for all $a \in A$. Therefore when we let $h \to 0$ we recover that $v$ is a viscosity subsolution of (3.6)

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0) + \sup_{a\in A}\{-L(t_0,x_0,a) - \nabla\varphi(t_0,x_0)\cdot f(t_0,x_0,a)\} \le 0.$$

It remains to prove that $v$ is a viscosity super-solution. Let $(t_0,x_0) \in Q_0$ and $\varphi \in C^\infty(\overline{Q_0})$ satisfy

$$0 = (v-\varphi)(t_0,x_0) = \min\left\{(v-\varphi)(t,x) : (t,x) \in \overline{Q_0}\right\}. \tag{3.10}$$

so that $v(t_0,x_0) = \varphi(t_0,x_0)$ and $v(t,x) \ge \varphi(t,x)$. In view of the definition of viscosity supersolution we need to show that

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0) + \sup_{a\in A}\{-L(t_0,x_0,a) - \nabla\varphi(t_0,x_0)\cdot f(t_0,x_0,a)\} \ge 0.$$

Similar to subsolution argument we obtain for $0 < h << 1$ with $t_0 + h < T$

$$\varphi(t_0,x_0) \ge \inf_{\alpha\in L^\infty([0,T],A)}\left\{\int_t^{t+h} L(s,X(s),\alpha(s))ds + \varphi(t_0+h,X(t_0+h))\right\}.$$

However, we are not in a position to choose a constant control and follow the sub-solution argument above, but we can work with $\epsilon$-optimal controls. In particular, choose $\alpha^h \in L^\infty([0,T],A)$ so that

$$\varphi(t_0,x_0) \ge \int_{t_0}^{t_0+h} L(u,X^h(u),\alpha^h(u))du + \varphi(t_0+h,X^h(t_0+h)) - h^2.$$

Using an integral form of Taylor expansion as above, cancelling $\varphi(t_0,x_0)$ terms and dividing by $h$ we get that

$$0 \ge \frac{1}{h}\int_{t_0}^{t_0+h}\left\{L(u,X^h(u),\alpha^h(u)) + \frac{\partial}{\partial t}\varphi(u,X^h(u)) + \nabla\varphi(u,X^h(u))\cdot f(u,X^h(u),\alpha^h(u))\right\}du.$$

The problem we face now is that we do not know if the uniform continuity of $\alpha^h(\cdot)$. Nevertheless, we can get

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0) \ge \limsup_{h\downarrow 0}\left\{\frac{1}{h}\int_{t_0}^{t_0+h} L(t_0,x_0,\alpha^h(u))du + \nabla\varphi(t_0,x_0)\cdot\frac{1}{h}\int_{t_0}^{t_0+h} f(t_0,x_0,\alpha^h(u))du\right\}.$$

Observe that if we set

$$\begin{aligned} F &= \{(L(t_0,x_0,a), f(t_0,x_0,a)) : a \in A\} \\ H(t,x,p) &= \sup\{-L(t,x,p) - f(t,x,p)\cdot p : (L,f) \in F\}. \end{aligned}$$

Therefore,

$$H(t,x,p) = \sup\left\{-L(t,x,p) - f(t,x,p)\cdot p : (L,f) \in \overline{\mathrm{co}(F)}\right\},$$

where the convex hull $\mathrm{co}(F)$ of $F$ is defined as

$$\mathrm{co}(F) = \left\{ \sum_{i=1}^{N} \lambda_i (L_i, f_i) : (L_i, f_i) \in F,\ 0 \le \lambda_i \le 1, \sum_{i=1}^{N} \lambda_i = 1 \text{ for } 1 \le i \le N \text{ with } N \in \mathbf{N} \right\}$$

We further note that

$$\left( \frac{1}{h} \int_{t_0}^{t_0+h} L(t_0, x_0, \alpha^h(u))du, \frac{1}{h} \int_{t_0}^{t_0+h} f(t_0, x_0, \alpha^h(u))du \right) \in \overline{\mathrm{co}(F)},$$

because, for instance, the Riemann sums approximating

$$\frac{1}{h} \int_{t_0}^{t_0+h} L(t_0, x_0, \alpha^h(u))du$$

can be expressed as convex combination of elements of $F$. Since the Riemann sums are in $\mathrm{co}(F)$, their limit $\frac{1}{h}\int_{t_0}^{t_0+h} L(t_0, x_0, \alpha^h(u))du$ lies in the closure of $\mathrm{co}(F)$. □

Next we want to justify our assumption that the value function $v$ of the dynamic programming equation is continuous.

**Theorem 14.** *Let $X^{\alpha}_{t,x}$ be the state process governed by the dynamics (3.5). The controls $\alpha$ take values in a compact set $A \subset \mathbf{R}^N$ and belong to the class $L^{\infty}([0,T]; A)$. The value function is given by*

$$v(t,x) = \inf_{\alpha \in L^{\infty}([0,T];A)} \left\{ \int_t^T L(u, X(u), \alpha(u))du + g(X(T)) \right\}$$

*$f$ is uniformly Lipschtiz continuous (1.2) and grows at most linearly (1.3). Moreover, $L$ is also uniformly Lipschtiz continuous (1.2) and $g$ satisfies the Lipschitz condition. Then $v$ is Lipschitz continuous.*

**Proof.** Denote the cost functional

$$J(t,x,\alpha) = \int_t^T L(u, X(u), \alpha(u))du + g(X(T)).$$

Then

$$\begin{aligned} |v(t,x) - v(t,y)| &= |\inf_{\alpha} J(t,x,\alpha) - \inf_{\alpha} J(t,y,\alpha)| \\ &\le \sup_{\alpha} \Big| J(t,x,\alpha) - J(t,y,\alpha) \Big| \\ &\le \sup_{\alpha} \left\{ \int_t^T K_L |X^{\alpha}_{t,x}(u) - X^{\alpha}_{t,y}(u)|du + K_g |X^{\alpha}_{t,x}(T) - X^{\alpha}_{t,y}(T)| \right\} \end{aligned}$$

for the Lipschitz constants $K_L$ and $K_g$ of $L$ and $g$ respectively. Then let

$$e_u = |X^{\alpha}_{t,x}(u) - X^{\alpha}_{t,y}(u)| \le |x-y| + \int_t^u K_f |X^{\alpha}_{t,x}(s) - X^{\alpha}_{t,y}(s)|dx$$

for the Lipschitz constant $K_f$ of $f$. Therefore applying Grönwall to

$$e_u \leq |x-y| + K_f \int_t^u e(s)ds$$

we get

$$|X^\alpha_{t,x}(u) - X^\alpha_{t,y}(u)| \leq |x-y| \exp\left(K_f(u-t)\right).$$

Hence,

$$|v(t,x) - v(t,y)| \leq \underbrace{\left\{ \int_t^T K_L \exp\left(K_f(u-t)\right) du + K_g \exp\left(K_f(T-t)\right) \right\}}_{\text{constant}} |x-y|.$$

□

### 3.2.2 Definition of Discontinuous Viscosity Solutions

Consider the Hamiltonian

$$H(t,x,u(x),\nabla u(x)) = 0 \quad x \in \mathcal{O} \subset \mathbf{R}^d,\ t \in [0,T) \tag{3.11}$$

for some $\mathcal{O}$ open but not necessarily bounded set. Define $Q = [0,T) \times \mathcal{O}$. We assume that $H$ is continuous in all of its components. An example of a Hamiltonian would be the dynamic programming equation

$$-\frac{\partial}{\partial t} v(t,x) + \sup_{a \in A} \left\{ -\nabla v(t,x) \cdot f(t,x,a) - L(t,x,a) \right\} = 0, \quad \forall t \in [0,T), \quad x \in \mathbf{R}^d.$$

In Section (3.2.1) we defined the viscosity solution of the dynamic programming equation for continuous functions. Now we extend this definition for general Hamiltonians and functions that we do not know a priori to be continuous. Define $v^*$ to be the upper-semicontinuous envelope of $v$, i.e.

$$v^*(t,x) = \lim_{\epsilon \downarrow 0} \sup_{(t',x') \in B_\epsilon(t,x)} v(t',x'). \tag{3.12}$$

Similarly the lower-semicontinuous envelope of $v$

$$v_*(t,x) = \lim_{\epsilon \downarrow 0} \inf_{(t',x') \in B_\epsilon(t,x)} v(t',x'). \tag{3.13}$$

**Definition 4. Viscosity Sub-solution**
$v \in L^\infty_{loc}(Q)$ is a viscosity sub-solution of (3.11) if for all $(t_0,x_0) \in Q$ and $\varphi \in C^\infty(\overline{Q})$ satisfying

$$0 = (v^* - \varphi)(t_0,x_0) = \max_{\overline{Q}} \left\{ (v^* - \varphi)(t,x) \right\}$$

we have

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0) + H(t_0,x_0,\varphi(t_0,x_0),\nabla\varphi(t_0,x_0)) \leq 0$$

**Definition 5. Viscosity Super-solution**
$v \in L^\infty_{loc}(Q)$ is a viscosity super-solution of (3.11) if for all $(t_0, x_0) \in Q$ and $\varphi \in C^\infty(\overline{Q})$ satisfying

$$0 = (v_* - \varphi)(t_0, x_0) = \min_{\overline{Q}} \{(v_* - \varphi)(t, x)\}$$

we have

$$-\frac{\partial}{\partial t}\varphi(t_0, x_0) + H(t_0, x_0, \varphi(t_0, x_0), \nabla\varphi(t_0, x_0)) \leq 0$$

Now we will start addressing the main issues in the theory of viscosity solutions, namely consistency, stability and uniquness and comparison.

### 3.2.3 Consistency

**Lemma 1. Consistency**:
$v \in C^1(Q)$ is a viscosity solution of (3.11) if and only if it is a classical solution of (3.11).

**Proof.** First, assume that $v \in C^1(Q)$ be a classical sub-solution of (3.11). Let $(t_0, x_0) \in [0, T) \times \mathcal{O}$ and $\varphi \in C^\infty(\overline{Q})$ satisfy

$$0 = (v - \varphi)(t_0, x_0) = \max_{\overline{Q}}(v - \varphi)(t, x).$$

Then since $v \in C^1(Q)$ we have $\nabla v(t_0, x_0) = \nabla\varphi(t_0, x_0)$ and $\frac{\partial}{\partial t}v(t_0, x_0) \leq \frac{\partial}{\partial t}\varphi(t_0, x_0)$, where equality holds for $t_0 \neq 0$. Therefore,

$$\begin{aligned} 0 &\geq -\frac{\partial}{\partial t}v(t_0, x_0) + H(t_0, x_0, v(x_0), \nabla v(t_0, x_0)) \\ &\geq -\frac{\partial}{\partial t}\varphi(t_0, x_0) + H(t_0, x_0, \varphi(x_0), \nabla\varphi(t_0, x_0)). \end{aligned}$$

This establishes that $v$ is a viscosity sub-solution. For the converse, if $v \in C^1(Q)$ is a viscosity sub-solution, choose as the test function $\varphi = v$ itself. Here the difficulty is that $v$ is not necessarily $C^\infty$ as the definition requires. However, this is tackled by using an equivalent definition of viscosity subsolution proved in the appendix, which involves taking test functions that are $C^1$. □

### 3.2.4 An example for an exit time problem

Let $\mathcal{O}$ be an open set in $\mathbf{R}^d$. Also let $\tau$ be the exit time of $X(t)$ from $\mathcal{O}$, where

$$X(t) = x + \int_0^t \alpha(s)ds.$$

The aim is to minimize

$$v(x) = \inf_\alpha \int_0^\tau \frac{1}{2}(1 + |\alpha(u)|^2)du.$$

The corresponding dynamic programming equation is given by

$$\begin{aligned} |\nabla v(x)|^2 &= 1 \quad \forall x \in \mathcal{O} \\ v(x) &= 0 \quad \forall x \notin \mathcal{O}. \end{aligned}$$

The solution to the dynamic programming equation is

$$v(x) = \text{dist}(x, \partial\mathcal{O}).$$

We claim that if $\mathcal{O} = (-1, 1)$, then $v(x) = 1-|x|$ is a viscosity solution. First we establish the sub-solution property. Let $x_0 \in (-1, 1)$ and $\varphi \in C^\infty(-1, 1)$ satisfy

$$0 = (v - \varphi)(x_0) = \max_{[-1,1]}(v - \varphi)(x). \tag{3.14}$$

If $x_0 \neq 0$, $\varphi$ touches the graph of $v$ at a point where the slope is either $-1$ or 1 so that $|\nabla\varphi(x_0)|^2 = 1$. For otherwise $x_0 = 0$ and according to (3.14) $\varphi$ is above the graph $v$ and has a derivative $-1 \leq \nabla\varphi(x_0) \leq 1$. We conclude it is a subsolution. For the viscosity super-solution let $x_0 \in (-1, 1)$ and $\varphi \in C^\infty(-1, 1)$ satisfy

$$0 = (v - \varphi)(x_0) = \min_{[-1,1]}(v - \varphi)(x). \tag{3.15}$$

then $x_0 \neq 0$ and $|\nabla\varphi(x_0)|^2 = 1$. $x_0 \neq 0$ because to satisfy the super-solution property at 0 we should have $|\nabla\varphi(0)|^2 \geq 1$, but then it is not possible for the test function $\varphi$ greater than $v$. Therefore, it is a viscosity solution.

It is in fact the unique viscosity solution. To illustrate the idea, we will show that $w(x) = |x| - 1$ is not a viscosity supersolution of $|\nabla v(x)|^2 = 1$. In particular, if we choose $x_0 = 0$ and $\tilde{\varphi}(x) = 1$, it clearly satisfies

$$0 = (w - \tilde{\varphi})(0) = \min_{[-1,1]}(w - \tilde{\varphi})(x).$$

Nevertheless, $\nabla\tilde{\varphi}(0) = 0 < 1$.

### 3.2.5 Comparison Theorem

In this section we consider a time homogeneous Hamiltonian and establish the first and simplest comparison theorem for it. The comparison result tells us if a viscosity sub-solution is smaller than a viscosity super-solution on the boundary $\partial\mathcal{O}$ of an open set $\mathcal{O}$, then this property is conserved over the whole closure $\overline{\mathcal{O}}$.

**Theorem 15.** ***Comparison Theorem:***
*Let $\mathcal{O}$ be an open bounded set. Consider the time-homogeneous Hamiltonian*

$$H(x, u(x), \nabla u(x)) = 0 \quad \forall x \in \mathcal{O}$$

*satisfying the property*

$$H\left(x, u, \frac{x-y}{\epsilon}\right) - H\left(y, v, \frac{x-y}{\epsilon}\right) \geq \beta(u-v) - K_1|x-y| - K_2\frac{|x-y|^2}{\epsilon} \tag{3.16}$$

*for some $\beta, K_1, K_2 \geq 0$. Let $u^*$ be a bounded viscosity sub-solution and $v_*$ a bounded viscosity super-solution satisfying $u^*(x) \leq v_*(x)$ on the boundary $\partial\mathcal{O}$. Then $u^*(x) \leq v_*(x)$ for all $x \in \overline{\mathcal{O}}$.*

*Remark.* The dynamic programming equation for infinite horizon deterministic control problems

$$H(x, u, p) = \beta u + \sup_{a\in A}\{-L(x, a) - f(x, a) \cdot p\}$$

satisfy the property (3.16) of the above theorem, provided that $L$ and $f$ are Lipschitz in $x$ and bounded.

**Proof.** Consider the auxiliary function

$$\varphi^\epsilon(x,y) = u^*(x) - v_*(y) - \frac{|x-y|^2}{2\epsilon}. \tag{3.17}$$

The auxiliary function $\varphi^\epsilon$ is upper-semicontinuous. So it assumes its maximum over the compact set $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$ at $(x^\epsilon, y^\epsilon)$. Since $(x^\epsilon, y^\epsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$, for sufficiently small $\epsilon$ either $x^\epsilon$ and $y^\epsilon$ belong to $\mathcal{O}$ or one or both of them are on the boundary.

The function $x \mapsto \varphi^\epsilon(x, y^\epsilon)$ is maximized at $x^\epsilon$. So if $x^\epsilon \in \mathcal{O}$ using as the test function $\varphi = v_*(y^\epsilon) + \frac{|x-y^\epsilon|^2}{2\epsilon}$ we get from the sub-solution property

$$0 \geq H\left(x^\epsilon, u^*(x^\epsilon), \frac{x^\epsilon - y^\epsilon}{\epsilon}\right).$$

Similarly $y \mapsto v_*(y) - \left(-\frac{|x^\epsilon - y|^2}{2\epsilon}\right)$ is minimized at $y^\epsilon$ so that if $y^\epsilon \in \mathcal{O}$ then

$$0 \leq H\left(y^\epsilon, v_*(y_\epsilon), \frac{x^\epsilon - y^\epsilon}{\epsilon}\right).$$

So consider the case when $x^\epsilon$ and $y^\epsilon$ belong to $\mathcal{O}$ for sufficiently small $\epsilon$. Set $p^\epsilon = \frac{x^\epsilon - y^\epsilon}{\epsilon}$. By (3.16)

$$\begin{aligned} 0 &\geq H(x^\epsilon, u^*(x^\epsilon), p^\epsilon) - H(y^\epsilon, v_*(y^\epsilon), p^\epsilon) \\ &\geq \beta[u^*(x^\epsilon) - v_*(y^\epsilon)] - K_1|x^\epsilon - y^\epsilon| - K_2\frac{|x^\epsilon - y^\epsilon|^2}{\epsilon}. \end{aligned}$$

Now for all $x \in \overline{\mathcal{O}}$, from the above inequality

$$\begin{aligned} \varphi^\epsilon(x,x) = u^*(x) - v_*(x) &\leq \varphi^\epsilon(x^\epsilon, y^\epsilon) \\ &\leq u^*(x^\epsilon) - v_*(y^\epsilon) \leq \frac{K_1}{\beta}|x^\epsilon - y^\epsilon| + \frac{K_2}{\beta}\frac{|x^\epsilon - y^\epsilon|^2}{\epsilon}. \end{aligned}$$

If we can establish that as $\epsilon \to 0$ we have $|x^\epsilon - y^\epsilon| \to 0$ as well as $\frac{|x^\epsilon - y^\epsilon|^2}{\epsilon} \to 0$, then we can conclude $\varphi^\epsilon(x,x) = u^*(x) - v_*(x) \leq 0$. We claim that $x^\epsilon, y^\epsilon$ converge to $x_0 \in \overline{\mathcal{O}}$ as $\epsilon \to 0$. Suppose without loss of generality $\varphi^\epsilon(x^\epsilon, y^\epsilon) \geq 0$. Then

$$\begin{aligned} \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon} &\leq u^*(x^\epsilon) - v_*(y^\epsilon) \\ &\leq \sup_{\overline{\mathcal{O}}}|u^*(x)| + \sup_{\overline{\mathcal{O}}}|v_*(x)| \leq C \\ &\Rightarrow \quad |x^\epsilon - y^\epsilon| \leq K\sqrt{\epsilon}. \end{aligned}$$

The next claim is that $u^*(x^\epsilon) - v_*(y^\epsilon) \to u^*(x_0) - v_*(x_0)$. By upper-semcontinuity

$$\begin{aligned} \limsup_{\epsilon \downarrow 0} u^*(x^\epsilon) &\leq u^*(x_0) \\ \liminf_{\epsilon \downarrow 0} -v_*(y^\epsilon) &\leq -v_*(x_0). \end{aligned}$$

This implies that

$$\limsup_{\epsilon \downarrow 0} u^*(x^\epsilon) - v_*(y^\epsilon) \leq (u^* - v_*)(x_0).$$

On the other hand,

$$\begin{aligned}\varphi^\epsilon(x_0,x_0) = u^*(x_0) - v_*(x_0) &\le u^*(x^\epsilon) - v_*(y^\epsilon) - \frac{|x^\epsilon - y^\epsilon|^2}{2\epsilon} \\ &\le u^*(x^\epsilon) - v_*(y^\epsilon),\end{aligned} \tag{3.18}$$

which implies that

$$(u^* - v_*)(x_0) \le \liminf_{\epsilon\downarrow 0} u^*(x^\epsilon) - v_*(y^\epsilon).$$

This establishes the claim. Moreover from (3.18) we have that

$$\frac{|x^\epsilon - y^\epsilon|^2}{\epsilon} \le u^*(x^\epsilon) - v_*(y^\epsilon) - (u^* - v_*)(x_0) \to 0$$

proving the first case. We show the second and the third case together, when for sufficiently small $\epsilon$ either $x^\epsilon$ or $y^\epsilon$ or both belong to $\partial\mathcal{O}$. Then from the closedness of the boundary and the assumption we get

$$u^*(x) - v_*(x) \le u^*(x^\epsilon) - v_*(y^\epsilon) \to u^*(x_0) - v_*(x_0) \le 0.$$

□

### 3.2.6 Stability

**Theorem 16.** ***Stability:***
*Suppose $u^n \in L^\infty_{loc}(\mathcal{O})$ is a viscosity sub-solution of*

$$H^n(x, u^n(x), \nabla u^n(x)) \le 0, \quad \forall x \in \mathcal{O}$$

*for all $n \in \mathbf{N}$. Assume that $H^n$ converges to $H$ uniformly. Define*

$$\bar{u}(x) = \limsup_{n\to\infty, x'\to x} u^n(x').$$

*Assume that $\bar{u}(x) < \infty$. Then $\bar{u}$ is a viscosity sub-solution of $H$.*

*Remark.* Similar statement holds for viscosity super-solutions.

**Proof.** Let $\varphi \in C^\infty$ and $x_0 \in \mathcal{O}$ be such that

$$0 = \max_{\overline{O}}(\bar{u} - \varphi)(x) = (\bar{u} - \varphi)(x_0).$$

We note that $\bar{u} = \limsup_{n\to\infty, x'\to x} u^{n,*}(x')$. By the definition of $\bar{u}$, we can find a sequence $\{x_n\} \in B(x_0, 1)$ such that $x_n \to x_0$ and

$$(\bar{u} - \varphi)(x_0) = \lim_{n\to\infty} u^{n,*}(x_n) - \varphi(x_n)$$

as $n \to \infty$. However, since $u^{n,*}$ is an upper semicontinuous function it assumes its maximum on the compact interval $\overline{B(x_0, 1)}$

$$(\bar{u} - \varphi)(x_0) \le \lim_{n\to\infty} \max_{\overline{B(x_0,1)}} u^{n,*}(x) - \varphi(x) = \lim_{n\to\infty} u^{n,*}(\overline{x_n}) - \varphi(\overline{x_n})$$

for some $\overline{x_n} \in \overline{B(x_0, 1)}$. Since $\overline{x_n} \in \overline{B(x_0, 1)}$, there exists a sequence by passing to a subsequence if necessary such that $\overline{x_n} \to \bar{x}$ for some $\overline{x} \in \overline{B(x_0, 1)}$. Again by upper-semicontinuity we obtain

$$0 = (\bar{u} - \varphi)(x_0) = \lim_{n\to\infty} u^{n,*}(\overline{x_n}) - \varphi(\overline{x_n}) \leq \bar{u}(\bar{x}) - \varphi(\bar{x}) \leq \bar{u}(x_0) - \varphi(x_0).$$

This implies that $x_0 = \bar{x}$, since $x_0$ is a strict maximum and $u^{n,*}(\overline{x_n}) \to u^*(x_0)$. Moreover, since $u^n$ is a viscosity solution, by an equivalent characterization in the appendix,

$$H^n(\overline{x_n}, u^{n,*}(\overline{x_n}), \nabla\varphi(\overline{x_n})) \leq 0.$$

We can now conclude that

$$H(x_0, \bar{u}(x_0), \nabla\varphi(x_0)) \leq 0$$

because $(\overline{x_n}, u^{n,*}(\overline{x_n}), \nabla\varphi(\overline{x_n})) \to (x_0, \bar{u}(x_0), \nabla\varphi(x_0))$ as $n \to \infty$. □

### 3.2.7 Exit Probabilities and Large Deviations

In this subsection we exhibit an application of stability property to large deviations demonstrating the power of the viscosity solutions. This proof is due to Evans and Ishii, [5] on the problem inspired by Varadhan [17].

Let $\epsilon > 0$ and $\mathcal{O}$ be an open bounded set in $\mathbf{R}^d$ and consider the stochastic differential equation

$$\begin{aligned} dX_s^\epsilon &= \epsilon\sigma(X_s^\epsilon)dW_s, \quad s > 0 \\ X_0^\epsilon &= x. \end{aligned}$$

Set $\gamma = \sigma\sigma^t$. Assume that $\gamma \in C^2$ and uniformly elliptic, i.e. $\gamma \geq \theta I$ for some $\theta > 0$. Moreover, $\gamma$ and $D\gamma$ are bounded. We consider the exit time from $\mathcal{O}$

$$\tau_x^\epsilon = \inf\{s > 0 : X_s^\epsilon \notin \mathcal{O}\}.$$

Then define for $\lambda > 0$

$$u^\epsilon(x) = E\left(\exp(-\lambda\tau_x^\epsilon)\right).$$

We expect that according to large deviations theory that $u^\epsilon \downarrow 0$ exponentially fast and we are interested in finding the rate.

By Feynman-Kac formula the partial differential equation associated with $u^\epsilon(x)$ is given as

$$\begin{aligned} 0 &= \lambda u^\epsilon(x) - \frac{\epsilon^2}{2}\gamma(x) : D^2u^\epsilon(x) \quad \forall x \in \mathcal{O} \\ 1 &= u^\epsilon(x) \quad \forall x \in \partial\mathcal{O}, \end{aligned} \tag{3.19}$$

where we denote by $A : B = \sum_{i,j=1}^d A_{ij}B_{ij}$. We make the Hopf transformation to study

$$v^\epsilon(x) = -\epsilon\log\left(u^\epsilon(x)\right).$$

Therefore,

$$u^\epsilon(x) = \exp\left(-\frac{v^\epsilon(x)}{\epsilon}\right)$$

so that

$$D^2u^\epsilon(x) = -\frac{1}{\epsilon}D^2v^\epsilon(x)u^\epsilon(x) + \frac{1}{\epsilon^2}Dv^\epsilon(x)\otimes Dv^\epsilon(x)u^\epsilon(x).$$

By plugging the above expression into (3.19) we obtain

$$0 = -\lambda - \frac{\epsilon}{2}\gamma(x) : D^2v^\epsilon(x) + \frac{1}{2}\gamma(x)Dv^\epsilon(x)\cdot Dv^\epsilon(x), \quad \forall x\in\mathcal{O} \tag{3.20}$$

$$0 = v^\epsilon(x), \quad \forall x\in\partial\mathcal{O}. \tag{3.21}$$

Note that by verification

$$\frac{1}{2}\gamma(x)Dv^\epsilon(x)\cdot Dv^\epsilon(x) = \sup_{a\in\mathbf{R}^d}\left\{-a\cdot Dv^\epsilon(x) - \frac{1}{2}\gamma^{-1}(x)a\cdot a\right\}.$$

Hence,

$$0 = -\lambda - \frac{\epsilon}{2}\gamma(x) : D^2v^\epsilon(x) + \sup_{a\in\mathbf{R}^d}\left\{-a\cdot Dv^\epsilon(x) - \frac{1}{2}\gamma^{-1}(x)a\cdot a\right\}, \quad \forall x\in\mathcal{O}$$

$$0 = v^\epsilon(x), \quad \forall x\in\partial\mathcal{O},$$

so that we can associate the PDE with the following control problem

$$dX_s^{\epsilon,\alpha} = \sqrt{\epsilon}\sigma(X_s^{\epsilon,\alpha})dW_s + \alpha(s)ds, \quad s>0, \quad X_0^{\epsilon,\alpha} = x,$$

$$v(x) = \inf_{\alpha\in\mathcal{A}}\left\{E\left[\int_0^{\tau_x^{\epsilon,\alpha}}\frac{1}{2}\gamma^{-1}(X_t^{\epsilon,\alpha})\alpha(t)\cdot\alpha(t) + \lambda dt\right]\right\},$$

where $\tau_x^{\epsilon,\alpha}$ is the first exit time from the set $\mathcal{O}$ and $\mathcal{A} = L^\infty([0,\infty);\mathbf{R}^d)$.

**Theorem 17.** *$v^\epsilon(x)\to v(x) = \sqrt{2\lambda}\mathcal{I}(x)$ uniformly on $\overline{\mathcal{O}}$, where $\mathcal{I}(x)$ is the unique viscosity solution of*

$$\gamma(x)D\mathcal{I}(x)\cdot D\mathcal{I}(x) = 1, \quad x\in\mathcal{O},$$
$$\mathcal{I}(x) = 0 \quad x\in\partial\mathcal{O}.$$

*Remark.* If $\gamma\equiv I$, then $|D\mathcal{I}(x)|^2 = 1$ is the eikonal equation and its unique viscosity solution is $\mathcal{I}(x) = \mathrm{dist}(x,\partial\mathcal{O})$.

**Proof.** There are two alternative approaches to prove this theorem. In the first approach, we show that

$$\overline{v}(x) = \limsup_{\epsilon\downarrow 0, x'\to x} v^\epsilon(x')$$

is a viscosity sub-solution of the limiting PDE

$$-\lambda + \frac{1}{2}\gamma(x)Dv(x)\cdot Dv(x) = 0 \tag{3.22}$$

of (3.20) as $\epsilon\to 0$ and

$$\underline{v}(x) = \liminf_{\epsilon\downarrow 0, x'\to x} v^\epsilon(x')$$

is a viscosity super-solution of the same PDE (3.22). Then by a comparison result, we can establish that $v^\epsilon\to v$ locally uniformly as $\epsilon\to 0$.

The second approach is using uniform $L^\infty$ estimates on $v^\epsilon$ and its gradient $Dv^\epsilon$. Since $\mathcal{O}$ has smooth boundary, for all $x_0 \in \partial\mathcal{O}$, we can find $\bar{x}$ and $r > 0$ such that for all $x \in \mathcal{O}$, we have $|x - \bar{x}| > r$. Then set

$$w(x) = k\left[|x - \bar{x}| - r\right]$$

We want to show $w$ is a super-solution of (3.20) for all $\epsilon > 0$ for sufficiently large $k$. By differentiating

$$\begin{aligned} Dw(x) &= k\frac{x - \bar{x}}{|x - \bar{x}|}, \\ D^2w(x) &= \frac{k}{|x - \bar{x}|}I - \frac{k}{|x - \bar{x}|^3}(x - \bar{x}) \otimes (x - \bar{x}). \end{aligned}$$

We plug these expressions into the PDE (3.20),

$$\begin{aligned} &-\lambda - \frac{\epsilon}{2}\gamma(x) : D^2w(x) + \frac{1}{2}\gamma(x)Dw(x) \cdot Dw(x) \\ &\geq -\lambda - \frac{\epsilon}{2}\sum_{i=1}^d \gamma^{ii}(x)\frac{k}{|x - \bar{x}|} + \left(\frac{1}{2}k^2 - \frac{\epsilon}{2}\frac{k}{|x - \bar{x}|}\right)\sum_{i,j=1}^d \gamma^{ij}(x)\left(\frac{x^i - \bar{x}^i}{|x - \bar{x}|}\right)\left(\frac{x^j - \bar{x}^j}{|x - \bar{x}|}\right). \end{aligned}$$

Since by construction $|x - \bar{x}| > r$ we get

$$\geq -\lambda - \frac{\epsilon}{2}\sum_{i=1}^d \gamma^{ii}(x)\frac{k}{r} + \left(\frac{1}{2}k^2 - \frac{\epsilon}{2}\frac{k}{r}\right)\sum_{i,j=1}^d \gamma^{ij}(x)\left(\frac{x^i - \bar{x}^i}{|x - \bar{x}|}\right)\left(\frac{x^j - \bar{x}^j}{|x - \bar{x}|}\right).$$

Then we can choose $k$ large enough and for $\epsilon < 2kr$, we have

$$\geq -\lambda - \frac{\epsilon}{2}\sum_{i=1}^d \gamma^{ii}(x)\frac{k}{r} + \theta\frac{k^2}{4}\frac{|x - \bar{x}|^2}{|x - \bar{x}|^2} \geq -\lambda - ck + ck^2 \geq 0,$$

by the uniform ellipticity condition and the boundedness of $\gamma$. This concludes that $w$ is a super-solution, hence $w(x) \geq v^\epsilon(x)$ for all $x \in \overline{\mathcal{O}}$. It follows that for small $\epsilon$, $v^\epsilon$ is uniformly bounded. Morever, since we can do this construction for all $x_0 \in \partial\mathcal{O}$, for small $\epsilon_0 > 0$

$$\sup_{0<\epsilon<\epsilon_0} \|Dv^\epsilon\|_{L^\infty(\partial\mathcal{O})} \leq \|Dw\|_{L^\infty(\partial\mathcal{O})} = k < \infty.$$

It remains to prove that $Dv^\epsilon$ is uniformly bounded for all $x \in \overline{\mathcal{O}}$ and sufficiently small $\epsilon$. Set

$$z(x) = |Dv^\epsilon(x)|^2.$$

There exists $x_0$ such that

$$z(x_0) = \max_{\overline{\mathcal{O}}} z(x).$$

If $x_0 \in \partial\mathcal{O}$, we are done, so assume that $x_0 \in \mathcal{O}$. Since $x_0$ is the maximizer,

$$\begin{aligned} 0 = z_i(x_0) &= 2\sum_{k=1}^d v_k^\epsilon(x_0)v_{ki}^\epsilon(x_0), \\ 0 &\leq -\frac{\epsilon}{2}\sum_{i,j=1}^d \gamma^{ij}z_{ij}(x_0). \end{aligned}$$

The second claim follows because both $\gamma$ and $-D^2 z(x_0)$ are non-negative definite so that the trace of their product is non-negative. It is easy to see

$$z_{ij}(x_0) = 2\sum_{k=1}^{d} v^\epsilon_{ki}(x_0)v^\epsilon_{kj}(x_0) + 2\sum_{k=1}^{d} v^\epsilon_k(x_0)v^\epsilon_{kij}(x_0).$$

Because

$$\gamma^{ij}(x_0)v^\epsilon_k(x_0)v^\epsilon_{kij}(x_0) = v^\epsilon_k(x_0)\left(\gamma^{ij}(x_0)v^\epsilon_{ij}(x_0)\right)_k - v^\epsilon_k(x_0)\gamma^{ij}_k(x_0)v^\epsilon_{ij}(x_0),$$

we obtain

$$\begin{aligned}
0 &\le -\frac{\epsilon}{2}\sum_{i,j=1}^{d}\gamma^{ij}(x_0)z_{ij}(x_0)\\
&\le -\epsilon\sum_{i,j,k=1}^{d}\gamma^{ij}(x_0)v^\epsilon_{ik}(x_0)v^\epsilon_{jk}(x_0) + v^\epsilon_k(x_0)\left(\gamma^{ij}(x_0)v^\epsilon_{ij}(x_0)\right)_k\\
&\qquad - v^\epsilon_k(x_0)\gamma^{ij}_k(x_0)v^\epsilon_{ij}(x_0)
\end{aligned}$$

By the uniform ellipticity condition

$$\epsilon\theta|D^2v^\epsilon(x)|^2 \le \epsilon\sum_{k=1}^{d}\sum_{i,j=1}^{d}\gamma^{ij}(x_0)v^\epsilon_{ik}(x_0)v^\epsilon_{jk}(x_0)$$

and by the PDE

$$-\frac{\epsilon}{2}\gamma^{ij}(x_0)v^\epsilon_{ij}(x_0) = \lambda - \frac{1}{2}\sum_{i,j=1}^{d}\gamma^{ij}(x_0)v^\epsilon_i(x_0)v^\epsilon_j(x_0).$$

These two statements imply that

$$\epsilon\theta|D^2v^\epsilon(x_0)|^2 \le \sum_{i,j,k=1}^{d}\left[-v^\epsilon_k(x_0)\left(\gamma^{ij}(x_0)v^\epsilon_i(x_0)v^\epsilon_j(x_0)\right)_k + \epsilon v^\epsilon_k(x_0)\gamma^{ij}_k(x_0)v^\epsilon_{ij}(x_0)\right]$$

Note that

$$\begin{aligned}
&\sum_{k=1}^{d} v^\epsilon_k(x_0)\left(\gamma^{ij}(x_0)v^\epsilon_i(x_0)v^\epsilon_j(x_0)\right)_k\\
=&\sum_{k=1}^{d}\left[\gamma^{ij}(x_0)\left[v^\epsilon_k(x_0)v^\epsilon_{ik}(x_0)v^\epsilon_j(x_0) + v^\epsilon_k(x_0)v^\epsilon_{jk}(x_0)v^\epsilon_i(x_0)\right] + \gamma^{ij}_k(x_0)v^\epsilon_i(x_0)v^\epsilon_j(x_0)v^\epsilon_k(x_0)\right]\\
=&\sum_{k=1}^{d}\gamma^{ij}_k(x_0)v^\epsilon_i(x_0)v^\epsilon_j(x_0)v^\epsilon_k(x_0)
\end{aligned}$$

because $z_i(x_0) = z_j(x_0) = 0$. Therefore, by Hölder inequality and boundedness of $\gamma$ and $D\gamma$

$$\begin{aligned}
&\epsilon\theta|D^2v^\epsilon(x_0)|^2 \le \sum_{i,j,k=1}^{d}\left|\gamma^{ij}_k(x_0)v^\epsilon_i(x_0)v^\epsilon_j(x_0)v^\epsilon_k(x_0)\right| + \epsilon\sum_{i,j,k=1}^{d}\left|v^\epsilon_k(x_0)\gamma^{ij}_k(x_0)v^\epsilon_{ij}(x_0)\right|\\
&\le C|Dv^\epsilon(x_0)|^3 + \epsilon C|Dv^\epsilon(x_0)||D^2v^\epsilon(x_0)|.
\end{aligned}$$

By the fact that $ab \leq \frac{\lambda^2 a^2}{2} + \frac{b^2}{\lambda^2 2}$ for any $\lambda > 0$ and by interpolation

$$\frac{\epsilon\theta}{2}|D^2 v^\epsilon(x_0)|^2 \leq C\left(|Dv^\epsilon(x_0)|^3 + 1\right).$$

Uniform ellipticity condition and similar inequalities as above imply that

$$\begin{aligned} z^2(x_0) = |Dv^\epsilon(x_0)|^4 &\leq C + C\left|\frac{1}{2}\sum_{i,j=1}^{d}\gamma^{ij}(x_0)v_i^\epsilon(x_0)v_j^\epsilon(x_0) - \lambda\right|^2 \\ &\leq C + C\left|\frac{\epsilon}{2}\sum_{i,j=1}^{d}\gamma^{ij}(x_0)v_{ij}^\epsilon(x_0)\right|^2 \\ &\leq C + C\epsilon^2|D^2 v^\epsilon(x_0)|^2 \leq C + \epsilon C(|Dv^\epsilon(x_0)|^3 + 1). \end{aligned}$$

This shows that $|Dv^\epsilon(x_0)|$ is uniformly bounded for sufficiently small $\epsilon > 0$. Then by Arzela-Ascoli theorem, there exists a subsequence $\epsilon_k \to 0$ and $v \in C^1(\overline{\mathcal{O}})$ such that $v^{\epsilon_k} \to v$ uniformly on $\overline{\mathcal{O}}$. □

### 3.2.8 Generalized Boundary Conditions

We recall the exit time problem from Section (1.7.3). Let $\mathcal{O} = (0,1) \subset \mathbf{R}$ and $T = 1$. Given the control $\alpha$, the system dynamics $X$ satisfies

$$\dot{X}(s) = \alpha(s), \quad s > t, \quad \text{and } X(t) = x.$$

The value function is

$$v(t,x) = \inf_\alpha \left\{\int_t^{\tau_x^\alpha} \frac{1}{2}\alpha(u)^2 du + X_x^\alpha(\tau_x^\alpha)\right\}$$

with $\tau_x^\alpha$ is the exit time. Then the resulting dynamic programming equation is

$$-v_t + \frac{1}{2}|v_x|^2 = 0, \quad \forall x \in (0,1),\ t < 1. \tag{3.23}$$

with boundary data

$$v(T,x) = x, \quad \forall x \in [0,1], \quad v(t,0) = 0, \quad \forall t \leq 1, \quad v(t,1) \leq 1, \quad \forall t \leq 1. \tag{3.24}$$

We found in Section (1.7.3) that

$$v(t,x) = \begin{cases} x - \frac{1}{2}(1-t) & (1-t) \leq x \leq 1 \\ \frac{1}{2}\frac{x^2}{1-t} & 0 \leq x \leq (1-t) \end{cases} \tag{3.25}$$

is the solution of the dynamic programming equation, but

$$v(t,1) < 1 \quad \forall t < 1,$$

i.e. boundary data is not attained for $x = 1$. This motivates the need for defining weak boundary conditions.

Let us remember the state constraint problem presented in Section (1.7.4). An open domain $\mathcal{O} \subset \mathbf{R}^d$ is given together with the controlled process

$$\dot{X}(t) = f(X(t), \alpha(t)), \quad X(0) = x$$

where the control $\alpha$ is admissible, i.e $\alpha \in \mathcal{A}_x$ if and only if $X_x^\alpha(t) \in \overline{\mathcal{O}}$ for all $t \geq 0$. The state constraint value process $v_{sc}(x)$ satisfies

$$v_{sc}(x) = \inf_{\alpha \in \mathcal{A}_x} \left\{ \int_0^\infty \exp(-\beta t) L(X_x^\alpha(t), \alpha(t)) dt \right\}.$$

Since the controlled process $X$ is not allowed to leave the domain $\overline{\mathcal{O}}$, we can formulate the state constraint problem with weak boundary conditions that are equal to infinity on $\partial\mathcal{O}$, because the controlled process is not going to achieve them. We state the following theorem without proof.

**Theorem 18.** *Soner, H.M. [13].*
*$v_{sc}(x)$ is a viscosity solution of the dynamic programming equation*

$$\beta v_{sc}(x) + H(x, Dv_{sc}(x)) = 0 \quad \forall x \in \mathcal{O}, \tag{3.26}$$

*where*

$$H(x, p) = \sup_{a \in A} \{-f(x, a) \cdot p - L(x, a)\}.$$

*It is also a viscosity super-solution on $\overline{\mathcal{O}}$, i.e. if*

$$(v_* - \varphi)(x_0) = \min \left\{ (v_* - \varphi)(x) : x \in \overline{\mathcal{O}} \right\} = 0$$

*then*

$$\beta\varphi(x_0) + H(x_0, D\varphi(x_0)) \geq 0$$

*even if $x_0 \in \partial\mathcal{O}$.*

*Remark.* If $x_0 \in \mathcal{O}$, then $''Dv_*(x_0) = D\varphi(x_0)''$, but if $x_0 \in \partial\mathcal{O}$, then

$$''D\varphi(x_0) = Dv_*(x_0) + \lambda\vec{n}(x_0)''$$

for some $\lambda > 0$, where $\vec{n}(x_0)$ is the unit outward normal at $x_0$. So if $v_{sc}(x) \in C^1(\overline{\mathcal{O}})$, then the boundary condition is

$$\beta v_{sc}(x) + H(x, Dv_{sc}(x_0) + \lambda\vec{n}(x_0)) \geq 0 \quad \forall \lambda \geq 0.$$

**Definition 6.** $v$ is a viscosity sub-solution of the equation

$$H(x, v(x), Dv(x)) = 0 \quad \forall x \in \mathcal{O} \tag{3.27}$$

with the boundary condition

$$v(x) = g(x) \quad \forall x \in \partial\mathcal{O} \tag{3.28}$$

whenever the smooth function $\varphi$ and the point $x_0 \in \overline{\mathcal{O}}$ satisfy

$$(v^* - \varphi)(x_0) = \max_{\overline{\mathcal{O}}} (v^* - \varphi)(x) = 0,$$

then either $x_0 \in \mathcal{O}$ and

$$H(x_0, \varphi(x_0), D\varphi(x_0)) \leq 0$$

or $x_0 \in \partial\mathcal{O}$ and

$$\min \{H(x_0, \varphi(x_0), D\varphi(x_0)), \varphi(x_0) - g(x_0)\} \leq 0.$$

**Definition 7.** $v$ is a viscosity super-solution of the equation (3.27) with the boundary condition (3.28) whenever the smooth function $\varphi$ and the point $x_0 \in \overline{\mathcal{O}}$ satisfy

$$(v^* - \varphi)(x_0) = \min_{\overline{\mathcal{O}}}(v^* - \varphi)(x) = 0,$$

then either $x_0 \in \mathcal{O}$ and

$$H(x_0, \varphi(x_0), D\varphi(x_0)) \geq 0$$

or $x_0 \in \partial\mathcal{O}$ and

$$\max\{H(x_0, \varphi(x_0), D\varphi(x_0)), \varphi(x_0) - g(x_0)\} \geq 0.$$

We return to the exit time problem presented in Section (1.7.3) and show $v(t,x)$ given by (3.25) is a viscosity solution of the corresponding dynamic programming equation (3.23) and the boundary condition (3.24). From previous analysis we only need to check the viscosity super-solution property for boundary points $x_0 = 1$ and smooth functions $\varphi$ satisfying

$$(v - \varphi)(t_0, 1) = \min_{\overline{\mathcal{O}}}(v - \varphi)(t, x) = 0,$$

since the viscosity sub-solution property is always satisfied, because $v(t,1) < g(x)$. Then we observe that

$$\varphi_t(t_0, 1) = v_t(t_0, 1) = \frac{1}{2} \quad \text{and} \quad \varphi_x(t_0, 1) \geq v_x(t_0, 1) = 1$$

so that the viscosity super-solution property follows

$$-\varphi_t(t_0, 1) + \frac{1}{2}\varphi_x^2(t_0, 1) \geq 0.$$

*Remark.* If the Hamiltonian $H$ is coming from a minimization problem, then we always satisfy $v \leq g$ on $\partial\mathcal{O}$, so only super-solution property on the boundary is relevant.

Next we prove a comparison result for the state constraint problem under that the assumption the supersolution is continuous on $\overline{\mathcal{O}}$.

**Theorem 19.** ***Comparison Result for the State Constraint Problem*** *Suppose $\mathcal{O}$ is open and bounded. Assume $v \in C(\overline{\mathcal{O}})$ is a viscosity super-solution of (3.26) on $\overline{\mathcal{O}}$ and $u \in L^\infty(\overline{O})$ is a viscosity sub-solution of (3.26) on $\mathcal{O}$. If there exists a non-decreasing continuous function $w : \mathbf{R}^+ \to \mathbf{R}^+$ with $w(0) = 0$ such that*

$$H(x,p) - H(y, p+q) \leq w(|x-y| + |x-y||p| + |q|) \tag{3.29}$$

*and $\mathcal{O}$ satisfies the interior cone condition, i.e. for all $x \in \overline{\mathcal{O}}$, there exists $h, r > 0$ and $\eta : \overline{\mathcal{O}} \to \mathbf{R}^d$ continuous such that*

$$B(x + t\eta(x), tr) \subset \mathcal{O} \quad \forall t \in [0, h], \tag{3.30}$$

*then $u^* \leq v$ on $\overline{\mathcal{O}}$.*

**Proof.** Let

$$(u^* - v)(z_0) = \max_{\overline{\mathcal{O}}}(u^* - v),$$

where $z_0 \in \overline{\mathcal{O}}$. We need to show that $(u^* - v)(z_0) \leq 0$. As it is a standard trick for comparison arguments, we define the auxiliary function $\phi^{\epsilon,\lambda}$ for $0 < \epsilon << 1$ and $\lambda > 0$ by

$$\phi^{\epsilon,\lambda}(x, y) = u^*(x) - v(y) - \left|\frac{x - y}{\epsilon} - \frac{2}{r}\eta(z_0)\right| - \frac{\lambda}{2}|y - z_0|^2.$$

By upper semi-continuity of $u^*$ and continuity of $v$, there exists $(x^\epsilon, y^\epsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ such that

$$\phi^{\epsilon,\lambda}(x^\epsilon, y^\epsilon) = \max_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}} \phi^{\epsilon,\lambda}(x, y).$$

By choosing $x = z_0 + \frac{2\epsilon}{r}\eta(z_0)$ and $y = z_0$

$$u^*\left(z_0 + \frac{2\epsilon}{r}\eta(z_0)\right) - v(z_0) = \phi^{\epsilon,\lambda}\left(z_0 + \frac{2\epsilon}{r}\eta(z_0), z_0\right) \leq \phi^{\epsilon,\lambda}(x^\epsilon, y^\epsilon). \quad (3.31)$$

This implies by the definition of $\phi^{\epsilon,\lambda}$ that

$$\left|\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right| + \frac{\lambda}{2}|y^\epsilon - z_0|^2 \leq 2\left[\|u^*\|_{L^\infty(\overline{\mathcal{O}})} + \|v\|_{L^\infty(\overline{\mathcal{O}})}\right] < \infty. \quad (3.32)$$

Since $x^\epsilon, y^\epsilon \in \overline{\mathcal{O}}$, they have a convergent subsequence to $z_1, z_2$ respectively. However, $z_1 = z_2$ for otherwise the left-hand-side of (3.32) would blow up as $\epsilon \to 0$. Now we claim that $z_1 = z_0$. If we let $\epsilon \downarrow 0$, from (3.31) we obtain

$$u^*(z_0) - v(z_0) + \limsup_{\epsilon\downarrow 0}\left|\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right| + \frac{\lambda}{2}|y^\epsilon - z_0|^2 \leq u^*(z_1) - v(z_1) \leq u^*(z_0) - v(z_0)$$

because of maximality of $u^* - v$ at $z_0$. Thus, we conclude that $z_0 = z_1$ and that

$$\limsup_{\epsilon\downarrow 0}\left|\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right| + \frac{\lambda}{2}|y^\epsilon - z_0|^2 = 0. \quad (3.33)$$

Express $x^\epsilon$ as

$$x^\epsilon = \left[y^\epsilon + \frac{2\epsilon}{r}\eta(y^\epsilon)\right] + \underbrace{\left[\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right]}_{\nu_\epsilon}\epsilon + \underbrace{\left[\frac{2}{r}\left(\eta(z_0) - \eta(y^\epsilon)\right)\right]}_{\mu_\epsilon}\epsilon. \quad (3.34)$$

We observe that both $|\nu_\epsilon|$ and $|\mu_\epsilon|$ converge to zero as $\epsilon \to 0$, because $\eta$ is continuous and by (3.33). Therefore, I can make $\epsilon(|\nu_\epsilon| + |\mu_\epsilon|) \leq 2\epsilon$ if $\epsilon << 1$. Then

$$x^\epsilon \in B\left(y^\epsilon + \frac{2\epsilon}{r}\eta(y^\epsilon), \epsilon(|\nu_\epsilon| + |\mu_\epsilon|)\right).$$

For $\epsilon$ sufficiently small we have $t = \frac{2\epsilon}{r} < h$ so that

$$x^\epsilon \in B\left(y^\epsilon + t\eta(y^\epsilon), rt\right) \subset\in \mathcal{O}.$$

Since we established $x^\epsilon \in \mathcal{O}$, the rest follows as the previos comparison argument. The map

$$x \mapsto u^*(x) - v(y^\epsilon) - \left|\frac{x - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right| - \frac{\lambda}{2}|y^\epsilon - z_0|^2$$

is maximized at $x^\epsilon$, hence

$$\beta u^*(x^\epsilon) + H(x^\epsilon, p^\epsilon) \leq 0$$

with

$$p^\epsilon = \frac{2}{\epsilon}\left(\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right).$$

Moreover, the map

$$y \mapsto u^*(x^\epsilon) - v(y) - \left|\frac{x^\epsilon - y}{\epsilon} - \frac{2}{r}\eta(z_0)\right| - \frac{\lambda}{2}|y - z_0|^2$$

is minimized at $y^\epsilon$ so that

$$\beta v(y^\epsilon) + H(y^\epsilon, p^\epsilon + q^\epsilon) \geq 0,$$

where

$$q^\epsilon = \lambda(z_0 - y^\epsilon).$$

By the assumption on the Hamiltonian it follows that

$$\beta(u^*(x^\epsilon) - v(y^\epsilon)) \leq H(y^\epsilon, p^\epsilon + q^\epsilon) - H(x^\epsilon, p^\epsilon) \leq w(|x^\epsilon - y^\epsilon| + |x^\epsilon - y^\epsilon||p^\epsilon| + |q^\epsilon|).$$

Because $|q^\epsilon|$ and $|x^\epsilon - y^\epsilon|$ converge to zero as $\epsilon \downarrow 0$, it is sufficient to prove that $|x^\epsilon - y^\epsilon||p^\epsilon| \to 0$.

$$\begin{aligned} |x^\epsilon - y^\epsilon||p^\epsilon| &\leq \left|x^\epsilon - y^\epsilon - \frac{2\epsilon}{r}\eta(z_0)\right||p^\epsilon| + \frac{2\epsilon}{r}|\eta(z_0)||p^\epsilon| \\ &\leq 2\left|\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right|^2 + \frac{4}{r}|\eta(z_0)|\left|\frac{x^\epsilon - y^\epsilon}{\epsilon} - \frac{2}{r}\eta(z_0)\right| \to 0 \quad \text{as} \quad \epsilon \to 0 \end{aligned}$$

We conclude that $u^*(z_0) - v(z_0) \leq 0$ □

*Remark.* In the case of $v$ discontinuous, we might not have a comparison theorem, as the next example illustrates.
Let $\mathcal{O} = \mathbf{R}^2 \backslash (\mathbf{R}^+ \times \mathbf{R}^+)$. Assume that the state dynamics is given as $\dot{X}(t) = (1, 0)$ and the cost function is $g(x_1, x_2) = x_2 + \frac{1}{1+x_1}$. Consider the value function

$$v(x) = \inf_\tau g(X(\tau)),$$

where $\tau$ is the exit time from $\mathcal{O}$. Then for $x_2 > 0$, $v(x) = x_2 + 1$ and for $x_2 \leq 0$, $v(x) = 0$. Hence

$$v^*(x_1, 0) = 1 \geq v_*(x_1, 0) = 0$$

Therefore, comparison theorem fails.

### 3.2.9 Unbounded Solutions on Unbounded Domains

We first consider a solution to the heat equation constructed by Tychonoff to demonstrate that we do not have uniqueness for the heat equation unless we impose a growth condition on the solution. The heat equation is given by

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x),\ \forall (t,x) \in \mathbf{R}^d \times (0,\infty) \\ u(0,x) &= g(x),\ \forall x \in \mathbf{R}^d. \end{aligned}$$

Define

$$u(t,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}, \quad g(t) = \exp\left(-t^{-\alpha}\right)$$

for some $1 < \alpha$ and $g^{(k)}(t)$ denotes the kth derivative of $g$. Formally,

$$\begin{aligned} u_t(t,x) &= \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{2(k-1)!} x^{2(k-1)}, \\ u_{xx}(t,x) &= \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!} (2k)(2k-1) x^{2(k-1)} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{2(k-1)!} x^{2(k-1)} \end{aligned}$$

so that $u$ is a solution of the heat equation.

We refer to the book of Fritz John to get the existence of $\theta = \theta(\alpha)$ with $0 < \theta$ such that for all $t \geq 0$

$$\left| g^{(k)}(t) \right| < \frac{k!}{(\theta t)^k} \exp\left(-\frac{1}{2} t^{-\alpha}\right).$$

Therefore, since $k!/(2k)! \leq 1/k!$ we obtain

$$\begin{aligned} |u(t,x)| &\leq \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{\theta t}\right)^k \right] \exp\left(-\frac{1}{2} t^{-\alpha}\right) \\ &= \exp\left(\frac{1}{t}\left(\frac{|x|^2}{\theta} - \frac{1}{2} t^{1-\alpha}\right)\right) := U(t,x) \end{aligned}$$

so that by comparison $u(t,x)$ converges uniformly for $t > 0$ and for bounded $x$. We can make a very similar analysis for $\sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{2(k-1)!} x^{2(k-1)}$ to show its uniform convergence for $t > 0$ and for bounded $x$. Therefore, we conclude that $u_t = u_{xx}$, where both $u_t$ and $u_{xx}$ are obtained by term by term differentiation.

This example illustrates that for uniqueness on unbounded domains one requires growth conditions. So the next question is how do the growth conditions affect the comparison results? We refer to a paper by Ishii [8] for the rest of the section.

If

$$\begin{aligned} u(x) + H(Du(x)) &\leq f(x) \quad \forall x \in \mathbf{R}^d, \\ v(x) + H(Dv(x)) &\geq g(x) \quad \forall x \in \mathbf{R}^d, \end{aligned}$$

when can I claim that

$$\sup_{x \in \mathbf{R}^d} (u - v)(x) \leq \sup_{x \in \mathbf{R}^d} (f - g)(x)?$$

A function $w$ in uniformly continuous on $\mathbf{R}^d$, denoted by $w \in UC(\mathbf{R}^d)$ if and only if there exists a modulus of continuity $m : \mathbf{R}^+ \to \mathbf{R}^+$ continuous and increasing, $m(0) = 0$ satisfying

$$|w(x) - w(y)| \leq m_w(|x - y|) \quad \forall x, y \in \mathbf{R}^d.$$

A function $w \in UC(\mathbf{R}^d)$ satisfies the linear growth condition, i.e. $|w(x)| \leq C(1 + |x|)$ as well as $|w(x) - w(y)| \leq C(1 + |x - y|)$. Next we prove the linear growth assumption for a uniformly continuous $w$ and the other fact follows similarly. Express

$$\begin{aligned} w(x) &= \sum_{k=1}^{\lfloor x \rfloor} \left\{ w\left(k\frac{x}{|x|}\right) - w\left((k-1)\frac{x}{|x|}\right)\right\} + w(x) - w\left(\frac{\lfloor x \rfloor}{|x|}x\right) \\ &\leq \sum_{k=1}^{\lfloor x \rfloor} m_w(1) + m_w\left(|x| - \frac{x}{|x|}\lfloor x \rfloor\right) \leq m_w(1)(1 + |x|). \end{aligned}$$

**Theorem 20.** ***Comparison Theorem***
*Assume that $H$ is continuous and $f, g$ are uniformly continuous on $\mathbf{R}^d$. If $u \in UC(\mathbf{R}^d)$ is a viscosity sub-solution of*

$$u(x) + H(Du(x)) = f(x) \quad x \in \mathbf{R}^d \tag{3.35}$$

*and $v \in UC(\mathbf{R}^d)$ is a viscosity super-solution of*

$$v(x) + H(Dv(x)) = g(x) \quad x \in \mathbf{R}^d, \tag{3.36}$$

*then comparison result,*

$$\sup_{x \in \mathbf{R}^d} (u - v)(x) \leq \sup_{x \in \mathbf{R}^d} (f - g)(x)$$

*holds.*

**Proof.** Without loss of generality we may assume that $\sup_{x \in \mathbf{R}^d}(f - g)(x) < \infty$.

The first step is to prove for every $\epsilon > 0$ the function

$$\phi(x, y) = u(x) - v(y) - \frac{1}{\epsilon}|x - y|^2$$

is bounded by above on $\mathbf{R}^d \times \mathbf{R}^d$. So fix $\epsilon > 0$ and introduce

$$\Phi(x, y) = u(x) - v(y) - \frac{1}{\epsilon}|x - y|^2 - \alpha(|x|^2 + |y|^2)$$

for some $\alpha > 0$. Since $u$ and $v$ are uniformly continuous, they satisfy the linear growth condition so that

$$\lim_{|x|+|y| \to \infty} \Phi(x, y) = -\infty.$$

This boundary condition along with the continuity of $\Phi$ asserts that there exists $(x_0, y_0) \in \mathbf{R}^{2d}$ where $\Phi$ attains its global maximum. In particular, $\Phi(x_0, x_0) \leq \Phi(x_0, y_0)$ and $\Phi(y_0, y_0) \leq \Phi(x_0, y_0)$. Therefore,

$$\Phi(x_0, x_0) + \Phi(y_0, y_0) \leq 2\Phi(x_0, y_0) \;\Rightarrow\; \frac{2}{\epsilon}|x_0 - y_0|^2 \leq u(x_0) - u(y_0) + v(x_0) - v(y_0).$$

Then uniform continuity of $u$ and $v$ imply that

$$\frac{2}{\epsilon}|x_0 - y_0|^2 \leq C(1 + |x_0 - y_0|).$$

Therefore by the inequality $ab \leq \frac{\lambda a^2}{2} + \frac{1}{2\lambda}b^2$

$$|x_0 - y_0|^2 \leq \epsilon C(1 + |x_0 - y_0|) \leq \epsilon C + \frac{1}{2}\epsilon^2 C^2 + \frac{1}{2}|x_0 - y_0|^2$$

which shows that

$$|x_0 - y_0| \leq C_\epsilon = (2\epsilon C + \epsilon^2 C^2)^{1/2}.$$

Now we take $x = y = 0$ and from $\Phi(0,0) \leq \Phi(x_0, y_0)$ observe that

$$\alpha(|x_0|^2 + |y_0|^2) \leq u(x_0) - u(0) + v(0) - v(y_0) \leq C(1 + |x_0| + |y_0|)$$

so that

$$\alpha^2(|x_0|^2 + |y_0|^2) \leq \alpha C + \alpha C|x_0| + \alpha C|y_0| \leq \alpha C + C^2 + \frac{1}{2}\alpha^2(|x_0|^2 + |y_0|^2).$$

Hence, for $0 < \alpha < 1$ we have $\alpha|x_0| \leq C_0$ and $\alpha|y_0| \leq C_0$. Then

$$x \to \Phi(x, y_0) = u(x) - [v(y_0) + \frac{1}{\epsilon}|x - y_0|^2 + \alpha|x|^2 + \alpha|y_0|^2]$$

is maximized at $x_0$. Since $u$ is a viscosity sub-solution,

$$u(x_0) + H\left(\frac{2}{\epsilon}(x_0 - y_0) + 2\alpha x_0\right) \leq f(x_0). \tag{3.37}$$

Moreover,

$$y \to -\Phi(x_0, y) = v(y) - [u(x_0) - \frac{1}{\epsilon}|x_0 - y|^2 - \alpha|x_0|^2 - \alpha|y|^2]$$

is minimized at $y_0$. Since $v$ is a viscosity super-solution,

$$v(y_0) + H\left(\frac{2}{\epsilon}(x_0 - y_0) - 2\alpha y_0\right) \geq g(x_0). \tag{3.38}$$

Subtracting (3.38) from (3.37) we obtain

$$\begin{aligned} u(x_0) - v(y_0) &\leq f(x_0) - g(x_0) + g(x_0) - g(y_0) + H\left(\frac{2}{\epsilon}(x_0 - y_0) - 2\alpha y_0\right) - H\left(\frac{2}{\epsilon}(x_0 - y_0) + 2\alpha x_0\right) \\ &\leq \sup_{x \in \mathbf{R}^d}(f - g) + w_g(|x_0 - y_0|) + b_H\left(\frac{2}{\epsilon}|x_0 - y_0| + 2\alpha|y_0|\right) + b_H\left(\frac{2}{\epsilon}|x_0 - y_0| + 2\alpha|x_0|\right), \end{aligned}$$

where $b_H(r) = \sup\{|H(p)| : |p| \leq r\}$. If we denote by $R_\epsilon = \frac{2}{\epsilon}C_\epsilon + 2C_0$, we get for $0 < \alpha < 1$

$$\begin{aligned} u(x) - v(y) - \frac{1}{\epsilon}|x - y|^2 - \alpha(|x|^2 + |y|^2) &\leq u(x_0) - v(y_0) - \frac{1}{\epsilon}|x - y|^2 - \alpha(|x|^2 + |y|^2) \\ &\leq u(x_0) - v(y_0) \leq \sup(f - g) + w_g(C_\epsilon) + 2b_H(R_\epsilon). \end{aligned}$$

Sending $\alpha \downarrow 0$, we recover that $\phi(x,y)$ is bounded above on $\mathbf{R}^{2d}$.

Let $\epsilon, \delta > 0$. Then there exists $(x_1, y_1) \in \mathbf{R}^{2d}$ such that

$$\left\{ \sup_{(x,y)\in\mathbf{R}^{2d}} u(x) - v(y) - \frac{1}{\epsilon}|x-y|^2 \right\} - \delta < u(x_1) - v(y_1) - \frac{1}{\epsilon}|x_1 - y_1|^2.$$

So take $\xi \in C^\infty(\mathbf{R}^{2d})$ such that $\xi(x_1, y_1) = 1$, $0 \le \xi(x,y) \le 1$ and $|D\xi| \le 1$. Moreover, the support of $\xi$ is in the ball $B((x_1,y_1),1)$. Set

$$\Phi^\delta(x,y) = \phi(x,y) + \delta\xi(x,y).$$

Then there exists $(x_2, y_2) \in B((x_1,y_1),1)$ such that $\Phi^\delta(x_2,y_2) = \max_{(x,y)\in\mathbf{R}^{2d}} \Phi^\delta(x,y)$. This fact follows because for $(x,y) \notin B((x_1,y_1),1)$,

$$\Phi^\delta(x_1,y_1) = \phi(x_1,y_1) + \delta \ge \phi(x,y) = \Phi^\delta(x,y).$$

Since $\Phi^\delta(x_2,x_2) \le \Phi^\delta(x_2,y_2)$,

$$u(x_2) - v(x_2) + \delta\xi(x_2,x_2) \le u(x_2) - v(y_2) - \frac{1}{\epsilon}|x_2 - y_2|^2 + \delta\xi(x_2,y_2).$$

Hence, by a similar analysis done earlier in the proof

$$\frac{1}{\epsilon}|x_2 - y_2|^2 \le C(1 + |x_2 - y_2|) + \delta \;\Rightarrow\; |x_2 - y_2| \le (2(C+\delta)\epsilon + \epsilon^2 C^2)^{1/2}.$$

Now

$$x \mapsto u(x) - \left[ v(y_2) + \frac{1}{\epsilon}|x - y_2|^2 + \delta\xi(x,y_2) \right]$$

is maximized at $x^2$, since $u$ is a viscosity sub-solution

$$u(x_2) + H\left( \frac{2}{\epsilon}(x_2 - y_2) + \delta D_x \xi(x_2,y_2) \right) \le f(x_2). \tag{3.39}$$

On the other hand

$$y \to v(y) - \left[ u(x) + \frac{1}{\epsilon}(x_2 - y_2) - \delta D_y \xi(x_2,y_2) \right]$$

is minimized at $y_2$. By the supersolution property of $v$,

$$v(y_2) + H\left( \frac{2}{\epsilon}(x_2 - y_2) - \delta D_y \xi(x_2,y_2) \right) \ge g(y_2). \tag{3.40}$$

Subtracting (3.40) from (3.39) and imitating the above arguments

$$u(x_2) - v(y_2) \le \sup_{x\in\mathbf{R}^d} (f-g)(x) + w_g(C_{\epsilon,\delta}) + w_{H,R}(2\delta),$$

where $C_{\epsilon,\delta} = (2(C+\delta)\epsilon + \epsilon^2 C^2)^{1/2}$, $R = \frac{2}{\epsilon}C_{\epsilon,\delta} + \delta$ and

$$w_{H,R}(r) = \sup\{|H(p) - H(q)| : p, q \in B(0,R), |p-q| < r\}$$

It follows that for all $x \in \mathbf{R}^d$

$$\begin{aligned} u(x) - v(x) &\le \Phi^\delta(x,x) \le \Phi^\delta(x_2,y_2) \le u(x_2) - v(y_2) + \delta \\ &\le \sup_{x\in\mathbf{R}^d} (f-g)(x) + w_g(C_{\epsilon,\delta}) + w_{H,R}(C_{\epsilon,\delta}) + \delta. \end{aligned}$$

Now for fixed $\epsilon > 0$, let $\delta \downarrow 0$ so that $C_{\epsilon,\delta} \to C_\epsilon = (2C\epsilon + \epsilon^2 C^2)^{1/2}$ and

$$u(x) - v(x) \leq \sup_{x \in \mathbf{R}^d} (f - g)(x) + w_g(C_\epsilon).$$

As $C_\epsilon \to 0$, as $\epsilon \downarrow 0$, we conclude that

$$\sup_{x \in \mathbf{R}^d} u(x) - v(x) \leq \sup_{x \in \mathbf{R}^d} f(x) - g(x).$$

□

Next we state some other comparison theorems without giving a proof. For a detailed discussion one can refer to [8].

For $\lambda > 0$ set

$$E_\lambda = \left\{ w \in C(\mathbf{R}^d) : \lim_{|x| \to \infty} w(t) e^{-\lambda |x|} = 0 \right\}.$$

**Theorem 21.** *Assume $H$ is Lipschitz with Lipschitz constant $A$, i.e. for $p, q \in \mathbf{R}^d$, $|H(p) - H(q)| \leq A|p - q|$ holds for some $A > 0$. Then if $f, g \in C(\mathbf{R}^d)$ and $u \in E_{1/A}$ is a viscosity subsolution of (3.35) and $v \in E_{1/A}$ is a viscosity supersolution of (3.36), then*

$$\sup_{x \in \mathbf{R}^d} (u(x) - v(x)) \leq \sup_{x \in \mathbf{R}^d} (f(x) - g(x)).$$

**Theorem 22.** *Assume $H$ is uniformly continuous on $\mathbf{R}^d$. Then if $f, g \in C(\mathbf{R}^d)$ and $u$ is a viscosity subsolution of (3.35) and $v$ is a viscosity supersolution of (3.36) satisfying $u, v \in \bigcap_{\lambda > 0} E_\lambda$ then*

$$\sup_{x \in \mathbf{R}^d} (u(x) - v(x)) \leq \sup_{x \in \mathbf{R}^d} (f(x) - g(x)).$$

For $m > 1$ we define

$$P_m(\mathbf{R}^d) = \left\{ w \in C(\mathbf{R}^d) : \sup_{x, y \in \mathbf{R}^d} \frac{|w(x) - w(y)|}{(1 + |x|^{m-1} + |y|^{m-1})|x - y|} < \infty \right\}.$$

**Theorem 23.** *Let $m > 1$ and $m^* = m/(m-1)$. Assume $H \in P_m(\mathbf{R}^d)$ and $f, g \in C(\mathbf{R}^d)$. If $u$ is a viscosity subsolution of (3.35) and $v$ is a viscosity supersolution of (3.36) satisfying $u, v \in \bigcup_{\mu < m^*} P_\mu(\mathbf{R}^d)$, then*

$$\sup_{x \in \mathbf{R}^d} (u(x) - v(x)) \leq \sup_{x \in \mathbf{R}^d} (f(x) - g(x)).$$

Consider the partial differential equation

$$u - |Du|^m = 0.$$

Observe that $u = 0$ is a solution to the above differential equation. On the other hand, $u(x) = (1/m^*)^{m^*} |x|^{m^*}$ is another solution of the partial differential equation for $m^* = \frac{m}{m-1}$. The function $u$ belongs to the class $P_{m^*}(\mathbf{R}^d)$ and the Hamiltonian $H(p) = -|p|^m$ is of class $P_m(\mathbf{R}^d)$. This example shows it is not possible to replace the condition $u, v \in \bigcup_{\mu < m^*} P_\mu(\mathbf{R}^d)$ by $u, v \in P_{m^*}(\mathbf{R}^d)$.

## 3.3 Viscosity Solutions for Stochastic Control Problems

Throughout the section we work with the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P}_0)$. Let $T > 0$. $\Omega = C_0[0,T]$ is the family of continuous functions on $[0,T]$ starting from 0. Let $\{\mathcal{F}_t^W\}_{0\le t\le T}$ be the filtration generated by a standard Brownian motion $W$ on $\Omega$. We take as the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ to be the completion of the right-continuous filtration $\{\mathcal{F}_{t^+}^W\}_{0\le t\le T}$. Also $\mathbf{P}_0$ denotes the Wiener measure. For a closed set $A \subset \mathbf{R}^d$, define the admissibility class $\mathcal{A} := \{\alpha \in L^\infty([0,T]\times\Omega; A) : \alpha_t \in \mathcal{F}_t\}$. The controlled state $X$ evolves according to the dynamics,

$$\begin{aligned} dX_s &= \mu(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s \\ X_t &= x. \end{aligned} \tag{3.41}$$

The coefficients $\mu$ and $\sigma$ are uniformly Lipschitz in $(t,x)$ and bounded. Furthermore $\gamma(t,x,a) = \sigma\sigma^T(t,x,a)$ is uniformly parabolic, i.e. for all $(t,x,a)$ there exists $c_0 > 0$ such that $\gamma(t,x,a) \ge c_o I > 0$. Denote by the solution of the stochastic differential equation (3.41) by $X_{t,x}^\alpha(\cdot)$. For an open set $\mathcal{O} \subset \mathbf{R}^d$, define $\tau_{t,x}^\alpha$ to be the exit time from $\mathcal{O}$,

$$\tau_{t,x}^\alpha = \inf\{t < s \le T : X_s \in \partial\mathcal{O},\ X_u \in \mathcal{O},\ \forall t \le u < s\} \wedge T.$$

Define the cost functional

$$J(t,x,\alpha) = E\left[\int_t^{\tau_{t,x}^\alpha} L(u, X_{t,x}^\alpha(u), \alpha(u))du + g\left(X_{t,x}^\alpha(\tau_{t,x}^\alpha)\right)\middle|\mathcal{F}_t\right],$$

where $L$ and $g$ are bounded below and $g$ is continuous. We are interested finding the value function

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} E[J(t,x,\alpha)].$$

### 3.3.1 Martingale Approach

For simplicity take $L \equiv 0$. Fix an initial datum $(0, X_0)$. We introduce the notation $X^\alpha = X_{0,X_0}^\alpha$ and $\tau^\alpha = \tau_{0,X_0}^\alpha$. For any $\alpha \in \mathcal{A}$ and $t \in [0,T]$ define

$$\mathcal{A}(t,\alpha) = \{\alpha' \in \mathcal{A} : \alpha'_u = \alpha_u\ \forall u \in [0,t]\ P_0 - a.s.\}.$$

and

$$\begin{aligned} Y_t^\alpha &:= \operatorname*{ess\,inf}_{\alpha'\in\mathcal{A}(t,\alpha)} J(t\wedge\tau^\alpha, X^\alpha(t\wedge\tau^\alpha), \alpha') \\ &= \operatorname*{ess\,inf}_{\alpha'\in\mathcal{A}(t,\alpha)} E\left[g(X^{\alpha'}(\tau^{\alpha'}))|\mathcal{F}_{t\wedge\tau^\alpha}\right]. \end{aligned}$$

Using the martingale approach we can characterize the optimal control $\alpha^*$ once we know it exists.

**Theorem 24.** ***Martingale Approach***

1. *For any $\alpha \in \mathcal{A}$, $Y^\alpha$ is a sub-martingale.*
2. *$\alpha^*$ is optimal if and only if $Y^* = Y^{\alpha^*}$ is a martingale.*

**Proof.** 1. First we note that $Y_t^\alpha = Y_{t\wedge\tau^\alpha}^\alpha$. Fix $s \geq t$. Since $E\left[g(X^{\alpha'}(\tau^{\alpha'}))|\mathcal{F}_t\right]$ is directed downward, there exist $\alpha_n \in \mathcal{A}(s,\alpha)$ such that

$$E\left[g(X^n(\tau^n))|\mathcal{F}_{s\wedge\tau^\alpha}\right] \downarrow Y_s^t \quad n \to \infty,$$

if we denote by $X^n = X^{\alpha_n}$ and $\tau^n = \tau^{\alpha^n}$. It follows that

$$\begin{aligned} E\left[Y_s^\alpha|\mathcal{F}_t\right] &= E\left[Y_{s\wedge\tau^\alpha}^\alpha|\mathcal{F}_t\right] = E\left[Y_{s\wedge\tau^\alpha}^\alpha|\mathcal{F}_{t\wedge\tau^\alpha}\right] \\ &= E\left[Y_s^\alpha|\mathcal{F}_{t\wedge\tau^\alpha}\right] = E\left[\lim_{n\to\infty} E\left[g(X^n(\tau^n))|\mathcal{F}_{s\wedge\tau^\alpha}\right]|\mathcal{F}_{t\wedge\tau^\alpha}\right] \end{aligned}$$

By monotone convergence theorem and the definition of $Y_t^\alpha$

$$= \lim_{n\to\infty} E\left[g(X^n(\tau^n))|\mathcal{F}_{t\wedge\tau^\alpha}\right] \geq Y_t^\alpha.$$

2. First assume that $Y^*$ is a martingale, then $Y_0^* = EY_T^* = E\left[g(X^*(\tau^*))\right]$. Since for any $\alpha \in \mathcal{A}$, $Y_0^* = Y_0^\alpha$, we get by the first part that

$$E\left[g(X^*(\tau^*))\right] = Y_0^\alpha \leq EY_T^\alpha = E\left[g(X^\alpha(\tau^\alpha))\right]$$

proving the optimality. Conversely, assume that $\alpha^*$ is optimal. Then

$$Y_0^* = \operatorname{ess\,inf}_{\alpha'} E\left[g(X^{\alpha'}(\tau^{\alpha'}))\right] = E\left[g(X^*(\tau^*))\right].$$

Since $\alpha^* \in \mathcal{A}(t,\alpha^*)$, $Y_t^* \leq E\left[g(X^*(\tau^*))|\mathcal{F}_{t\wedge\tau^*}\right]$, it follows that by first part

$$Y_0^* \leq E(Y_t^*) \leq E\left[g(X^*(\tau^*))\right] = Y_0^*$$

so that $Y_t^*$ is a submartingale with constant expectation, therefore it is a martingale.

□

### 3.3.2 Weak Dynamic Programming Principle

Next we consider the weak dynamic programming principle due to Bouchard and Touzi, [2].

**Theorem 25.** ***Weak Dynamic Programming Principle***
*Let $Q = [0,T) \times \mathcal{O}$ and $\overline{Q} = [0,T] \times \overline{\mathcal{O}}$.*

1. *If $\varphi : \overline{Q} \to \mathbf{R}$ is measurable and $v(t,x) \geq \varphi(t,x)$ for all $(t,x) \in \overline{Q}$ and $\theta$ is a stopping time, then*

$$v(t,x) \geq \inf_{\alpha\in\mathcal{A}} E\left[\varphi(\theta\wedge\tau_{t,x}^\alpha, X_{t,x}^\alpha(\theta\wedge\tau_{t,x}^\alpha))\right] \quad \forall (t,x) \in Q. \tag{3.42}$$

2. *If $\varphi(t,x)$ is continuous and $v(t,x) \leq \varphi(t,x)$ for all $(t,x) \in \overline{Q}$ and $\theta$ is a stopping time, then*

$$v(t,x) \leq \inf_{\alpha\in\mathcal{A}} E\left[\varphi(\theta\wedge\tau_{t,x}^\alpha, X_{t,x}^\alpha(\theta\wedge\tau_{t,x}^\alpha))\right] \quad \forall (t,x) \in Q. \tag{3.43}$$

*Remark.* If $v$ is continuous then the weak dynamic programming principle is equivalent to the classical dynamic programming principle.

Fix the notation $\tau^\alpha = \tau^\alpha_{t,x}$, $\eta^\alpha = \tau^\alpha \wedge \theta$ and $X^\alpha = X^\alpha_{t,x}$. Formally, we want to have

$$\begin{aligned} J(t,x,\alpha) &= E\left[g(X^\alpha(\tau^\alpha))|\mathcal{F}_t\right] = E\left[g(X^\alpha(\tau^\alpha))|\mathcal{F}_{t\wedge\tau^\alpha}\right] \\ &= E\left[g(X^\alpha_{\eta^\alpha,X^\alpha(\eta^\alpha)}(\tau^\alpha))|\mathcal{F}_{\eta^\alpha}\right] \\ &= J(\eta^\alpha, X^\alpha(\eta^\alpha),'\beta') \geq v(\eta^\alpha, X^\alpha(\eta^\alpha)). \end{aligned}$$

since $g(\cdot)$ is $\mathcal{F}(\tau^\alpha)$ measurable and $X^\alpha_{t,x}(\tau^\alpha) = X_{\eta^\alpha,X^\alpha(\eta^\alpha)}(\tau^\alpha)$. However, there are two major problems with the argument above. First, we do not know what the candidate control $'\beta'$ is. Since by definition $J(t,x,\alpha)$ is a random variable measurable to $\mathcal{F}_t$, we would expect $\beta$ to depend on $\bar\omega \in \Omega$ as well as on the time $t$. Moreover, we also do not know whether $v(\eta^\alpha, X^\alpha(\eta^\alpha))$ is measurable or not. To avoid these problems we replace $v(t,x)$ in the statement of the theorem by a measurable $\varphi(t,x)$ such that $\varphi(t,x) \leq v(t,x)$.

The proof of the dynamic programming principle relies on the fact that once we observe the realization $J(t,x,\alpha)(\bar\omega)$ for $\bar\omega \in \Omega$, then we know $\alpha$ until up to time $t$, because $\alpha$ is adapted. Therefore, we only care what $\alpha$ does after time $t$.

**Proof.** Define $\hat\Omega = C_0([0,T-t])$. Let $\hat W$ be the canonical Brownian Motion on $\hat\Omega$. As before, we take as the filtration $\{\hat{\mathcal{F}}_t\}_{0\leq t\leq T}$ to be the completion of the right-continuous filtration $\{\hat{\mathcal{F}}^{\hat W}_{t+}\}_{0\leq t\leq T}$. Set $\hat{\mathcal{A}} = \{\beta \in L^\infty([0,T-t]\times\hat\Omega; A) : \beta_t \in \hat{\mathcal{F}}_t\}$. Take $\hat{\mathbf{P}}_0$ to be the Wiener measure. For a control $\beta \in \hat{\mathcal{A}}$ consider the controlled state process $\hat X$ to evolve according to

$$\hat X_u = x + \int_0^u \mu(s+t, \hat X_s, \beta_s)ds + \int_0^u \sigma(s+t, \hat X_s, \beta_s)d\hat W_s \quad \forall u \in [0,T-t].$$

Denote the solution by $\hat X^\beta_{0,x}(\cdot)$. Also let

$$\begin{aligned} \hat J(t,x,\beta) &= E\left[g(\hat X^\beta_{0,x}(T-t))\right] \\ \hat v(t,x) &= \inf_{\beta\in\hat{\mathcal{A}}} \hat J(t,x,\beta). \end{aligned}$$

We will show that $v(t,x) = \hat v(t,x)$ and $J(t,x,\alpha) \geq \hat v(t,x)$ $\mathbf{P}_0$-almost surely. Given $\alpha \in \mathcal{A}$ and $\bar\omega \in \Omega$, define $\beta^{t,\bar\omega} \in L^\infty([0,T-t]\times\bar\Omega; A)$ by

$$\beta^{t,\bar\omega}_u(\hat\omega) := \alpha_{u+t}(\bar\omega \otimes \hat\omega).$$

$\beta$ depends on $\alpha$ and

$$(\bar\omega\otimes\hat\omega)_u = \begin{cases} \bar\omega(u) & 0 \leq u \leq t \\ \hat\omega(u-t) + \bar\omega(t) & t \leq u \leq T \end{cases}.$$

Clearly, $(\bar\omega\otimes\hat\omega) \in \Omega$. Then $\mathbf{P}_0$ almost surely,

$$J(t,x,\alpha)(\bar\omega) = \hat J(t,x,\beta^{t,\bar\omega}) \geq \hat v(t,x)$$

so that $v(t,x) \geq \hat v(t,x)$. Conversely, given $\beta \in \hat{\mathcal{A}}$ define $\alpha^t \in L^\infty([0,T]\times\Omega; A)$ by

$$\alpha^t_u(\omega) = \begin{cases} a_0 & 0 \leq u \leq t \\ \beta_{u-t}(\omega^t(\omega)) & t \leq u \leq T \end{cases},$$

where $a_0 \in \mathbf{R}$ is arbitrary and $(\omega^t(\omega))_s = \omega_{t+s} - \omega_t$. Then

$$\hat{J}(t,x,\beta) = J(t,x,\alpha^t) = E\left[J(t,x,\alpha^t)\right] \geq v(t,x).$$

This concludes the proof of $v(t,x) = \hat{v}(t,x)$. Using these claims, for any stopping time $\eta$ and $\mathcal{F}_\eta$-measurable $\xi$ we obtain

$$\begin{aligned} E\left[g(X^\alpha_{\eta,\xi}(\tau^\alpha))|\mathcal{F}_\eta\right](\bar{\omega}) &= \hat{J}(\eta(\bar{\omega}), \xi(\bar{\omega}), \beta^{\eta(\bar{\omega}),\bar{\omega},\alpha}) \\ &\geq \hat{v}(\eta,\xi) = v(\eta,\xi) \geq \varphi(\eta,\xi). \end{aligned}$$

Choose $(\eta,\xi) = (\eta^\alpha, X^\alpha(\eta^\alpha))$, take expectations on both sides and minimize over all $\alpha \in \mathcal{A}$. This finishes the proof of the first statement of weak dynamic programming principle.

For the second claim, we first show that for any $\alpha \in \mathcal{A}$ the map $(t,x) \mapsto E[J(t,x,\alpha)]$ is continuous for $(t,x) \in \overline{Q}$. So let $\alpha \in \mathcal{A}$ be arbitrary and $(t,x) \in \overline{Q}$. By continuity of $g$, it is sufficient to prove $(t,x) \to \tau^\alpha_{t,x}$ is continuous. Take any sequence $(t_n,x_n) \to (t,x)$. Then $\tau^\alpha_{t_n,x_n} \to \theta := \limsup_{n\to\infty} \tau^\alpha_{t_n,x_n}$. Since $X^\alpha_{t_n,x_n} \to X^\alpha_{t,x}$ as $n \to \infty$, either $X(\theta) \in \partial\mathcal{O}$ or $\theta = T$. Since $\sigma$ is non-degenerate, $\tau^\alpha_{t,x} = \theta$.

For any $\epsilon > 0$ and $(t,x) \in \overline{Q}$, there exists $\alpha^{t,x,\epsilon} \in L^\infty([0,T] \times \Omega; A)$ such that

$$E\left[J(t,x,\alpha^{t,x,\epsilon})\right] \leq \varphi(t,x) + \frac{1}{2}\epsilon.$$

Define the set

$$\mathcal{O}^{t,x,\epsilon} := \left\{(t',x') \in \overline{Q} : E\left[J(t',x',\alpha^{t,x,\epsilon})\right] < \varphi(t',x') + \epsilon\right\}.$$

By continuity of $E\left[J(t,x,\alpha^{t,x,\epsilon})\right]$ and $\varphi(t,x)$ in $(t,x)$, we can deduce that $\mathcal{O}^{t,x,\epsilon}$ is open in $\overline{Q}$. As $(t,x) \in \mathcal{O}^{t,x,\epsilon}$, $\{\mathcal{O}^{t,x,\epsilon}\}_{(t,x)\in\overline{Q}}$ is an open cover of $\overline{Q}$. If $\overline{Q}$ is bounded, we can extract a finite subcover by compactness. Otherwise $K_M = \overline{Q} \cap \overline{B(0,M)}$ is bounded, so we can have a finite subcover for any $M$. Since $\bigcup_M K_M = \overline{Q}$, there exists a countable family $(t_n,x_n)$ such that $\overline{Q} = \bigcup_{n=1}^\infty \mathcal{O}^{t_n,x_n,\epsilon}$.

Fix $\alpha$, $(t,x) \in \overline{Q}$ and a stopping time $\theta$. Define $\eta = \theta \wedge \tau^\alpha_{t,x}$. We want to consider

$$\alpha^\epsilon_u = \begin{cases} \alpha_u(\omega) & t \leq u \leq \eta(\omega) \\ \alpha^{\eta(\omega),X(\eta(\omega)),\epsilon} & \eta(\omega) \leq u \leq T \end{cases}$$

However, then we are faced with well-definedness and measurability problems, since the sets $\mathcal{O}^{t,x,\epsilon}$ may have nonempty intersection. So define

$$C_1 = \mathcal{O}^{t_1,x_1,\epsilon}, \ldots, C_{k+1} = \left[\mathcal{O}^{t_{k+1},x_{k+1},\epsilon}\right] \setminus \bigcup_{j=1}^k C_j$$

so that we still have $\bigcup_{k=1}^\infty C_k = \overline{Q}$. Now we set

$$\alpha^\epsilon_u = \begin{cases} \alpha_u(\omega) & t \leq u \leq \eta(\omega) \\ \alpha^{t_k,x_k,\epsilon}(\omega) & \eta(\omega) \leq u \leq T \text{ and } (\eta(\omega), X(\eta(\omega))) \in C_k \end{cases}.$$

By construction, the sets $C_k$ are disjoint and $\alpha^\epsilon_u \in \mathcal{A}$. Do the above construction with $\alpha = \alpha^n$ such that

$$E\left[\varphi(\theta \wedge \tau^{\alpha^n}_{t,x}, X^{\alpha^n}_{t,x}(\theta \wedge \tau^{\alpha^n}_{t,x}))\right] \leq \inf_{\alpha\in\mathcal{A}} E\left[\varphi(\theta \wedge \tau^\alpha_{t,x}, X^\alpha_{t,x}(\theta \wedge \tau^\alpha_{t,x}))\right] + \frac{1}{n}.$$

Then by the construction above for $\alpha^{\epsilon,n}$ we have

$$\begin{aligned}\frac{1}{n} + \inf_{\alpha\in\mathcal{A}} E\left[\varphi(\theta\wedge\tau^{\alpha}_{t,x}, X^{\alpha}_{t,x}(\theta\wedge\tau^{\alpha}_{t,x}))\right] &\geq E\left[J(\theta\wedge\tau^{\alpha^n}_{t,x}, X^{\alpha^n}_{t,x}(\theta\wedge\tau^{\alpha^n}_{t,x}), \alpha^{\epsilon,n})\right] - \epsilon \\ &\geq E(J(t,x,\alpha^{\epsilon,n})) - \epsilon \geq v(t,x) - \epsilon.\end{aligned}$$

Since the above statement holds for any $\epsilon > 0$ and for any $n$, we are done. □

### 3.3.3 Dynamic Programming Equation

As defined at the beginning of the section, we work with the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}, \mathbf{P}_0)$ and the controls $\alpha : [0,T]\times\Omega \to A \subseteq \mathbf{R}^N$ belong to the admissibility class

$$\mathcal{A} := \{\alpha \in L^{\infty}([0,T]\times\Omega; A) : \alpha \text{ is adapted to the filtration } \{\mathcal{F}_t\}\}.$$

Let $\mathcal{O} \subseteq \mathbf{R}^d$ be an open set. Consider the controlled diffusion

$$\begin{aligned} dX_s &= \mu(s, X_s, \alpha_s)ds + \sigma(s, X_s, \alpha_s)dW_s \\ X_t &= x \in \mathcal{O}.\end{aligned}$$

We denote the solution of this stochastic differential equation by $X^{\alpha}_{t,x}(\cdot)$. The coefficients $\mu$ and $\sigma$ are bounded and uniformly Lipschitz in $(t,x) \in [0,T]\times\mathcal{O}$. Moreover, $\gamma(t,x,a) := \sigma(t,x,a)\sigma(t,x,a)^T \geq -c_0 I$. $\tau^{\alpha}_{t,x}$ is the exit time of $X^{\alpha}_{t,x}(\cdot)$ from $\mathcal{O}$. Then the value function $v(t,x)$ is given by

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} E\left[\int_t^{\tau^{\alpha}_{t,x}\wedge T} \mathbf{\Lambda_u} L(u, X_u, \alpha_u)du + \mathbf{\Lambda}_{(\tau^{\alpha}_{\mathbf{t},\mathbf{x}}\wedge\mathbf{T})} g(\tau^{\alpha}_{t,x}\wedge T, X^{\alpha}_{t,x}(\tau^{\alpha}_{t,x}\wedge T))\right],$$

where

$$\mathbf{\Lambda_u} = \exp\left(-\int_t^u r(s, X_s, \alpha_s)ds\right).$$

We assume that $r$ is uniformly continuous in $(t,x)$, non-negative and bounded. Moreover $L$ is bounded, continuous in $t$ and uniformly Lipschitz in $x$. Suppose $g$ is continuous and bounded from below.

**Theorem 26.** ***Dynamic Programming Equation***

1. *$v(t,x)$ is a viscosity sub-solution of the dynamic programming equation*

$$-v_t(t,x) + H(t,x,v(t,x),\nabla v(t,x), D^2 v(t,x)) = 0 \quad (t,x)\in[0,T]\times\mathcal{O}, \tag{3.44}$$

*where*

$$H(t,x,v,p,\Gamma) = \sup_{a\in A}\left\{-L(t,x,a) - \mu(t,x,a)\cdot p - \frac{1}{2}tr(\gamma(t,x,a)\Gamma) + r(t,x,a)v(t,x)\right\},$$

*if for any smooth $\varphi$ satisfying $v \leq \varphi$ and stopping time $\theta$ we have*

$$v(t,x) \leq \inf_{\alpha\in\mathcal{A}} E\left[\int_t^{\tau^{\alpha}_{t,x}\wedge\theta} \mathbf{\Lambda_u} L(u, X^{\alpha}_{t,x}(u), \alpha(u))du + \mathbf{\Lambda}_{(\tau^{\alpha}_{\mathbf{t},\mathbf{x}}\wedge\theta)}\varphi(\tau^{\alpha}_{t,x}\wedge\theta, X^{\alpha}_{t,x}(\tau^{\alpha}_{t,x}\wedge\theta))\right]. \tag{3.45}$$

2. *$v(t,x)$ is a viscosity super-solution of the dynamic programming equation*

$$-v_t(t,x) + H(t,x,v(t,x),\nabla v(t,x),D^2v(t,x)) = 0 \quad (t,x) \in [0,T]\times\mathcal{O},$$

*if for any smooth $\varphi$ satisfying $v \geq \varphi$ and stopping time $\theta$ we have*

$$v(t,x) \geq \inf_{\alpha\in\mathcal{A}} E\left[\int_t^{\tau^\alpha_{t,x}\wedge\theta} \mathbf{\Lambda_u} L(u,X^\alpha_{t,x}(u),\alpha(u))du + \mathbf{\Lambda}_{(\tau^\alpha_{\mathbf{t},\mathbf{x}}\wedge\theta)}\varphi(\tau^\alpha_{t,x}\wedge\theta, X^\alpha_{t,x}(\tau^\alpha_{t,x}\wedge\theta))\right]. \tag{3.46}$$

**Proof.** To prove the subsolution property let $\varphi$ be a smooth function and the point $(t_0,x_0)\in[0,T)\times\mathcal{O}$ satisfy

$$(v^*-\varphi)(t_0,x_0) = \max\left\{(v^*-\varphi)(t,x) : (t,x)\in[0,T]\times\overline{\mathcal{O}}\right\} = 0.$$

In view of the definition of the viscosity sub-solution we need to show that

$$-\varphi_t(t_0,x_0) + H(t_0,x_0,\varphi(t_0,x_0),\nabla\varphi(t_0,x_0),D^2\varphi(t_0,x_0)) \leq 0.$$

Fix $\alpha_t = a\in\mathcal{A}$ and $\theta = t+\frac{1}{n}$. Then the dynamic programming principle (3.45) implies that for all $(t,x)\in[0,T)\times\mathcal{O}$

$$v(t,x) \leq E\left[\int_t^{\tau^\alpha_{t,x}\wedge t+\frac{1}{n}} \mathbf{\Lambda_u} L(u,X^\alpha_{t,x}(u),\alpha(u))du + \mathbf{\Lambda}_{(\tau^\alpha_{\mathbf{t},\mathbf{x}}\wedge\mathbf{t}+\frac{1}{\mathbf{n}})}\varphi(\tau^\alpha_{t,x}\wedge t+\frac{1}{n}, X^\alpha_{t,x}\left(\tau^\alpha_{t,x}\wedge t+\frac{1}{n})\right)\right].$$

Choose a sequence $(t_n,x_n)\to(t_0,x_0)$ as $n\to\infty$ such that $v(t_n,x_n)\to v^*(t_0,x_0) = \varphi(t_0,x_0)$ and

$$|v(t_n,x_n)-\varphi(t_n,x_n)| \leq |v(t_n,x_n)-v^*(t_0,x_0)| + |\varphi(t_0,x_0)-\varphi(t_n,x_n)| \leq \frac{1}{n^2}.$$

Starting at $(t_n,x_n)$ we use the control $\alpha_t = a$. Denote

$$\eta^n = \left(t_n+\frac{1}{n}\right)\wedge\tau^a_{t_n,x_n} \quad X^n = X^a_{t_n,x_n}.$$

Then

$$\begin{aligned}\varphi(t_n,x_n) &= v(t_n,x_n)+e_n\\ &\leq E\left[\int_{t_n}^{\eta^n} \mathbf{\Lambda_u} L(u,X^n(u),a)du + \mathbf{\Lambda}_{\eta^\mathbf{n}}\varphi(\eta^n,X^n(\eta^n))\right]+e_n, \end{aligned}\tag{3.47}$$

where $|e_n|\leq\frac{1}{n^2}$. Applying Ito's lemma to $\mathbf{\Lambda}_u\varphi(u,X^n(u))$ yields

$$\begin{aligned}d(\mathbf{\Lambda}_u\varphi(u,X^n(u))) = \mathbf{\Lambda_u}\Big(&-r(u,X^n(u),a)\varphi(u,X^n(u)) + \varphi_t(u,X^n(u)) + \nabla\varphi(u,X^n(u))\cdot\mu(u,X^n(u),a)\\ &+\frac{1}{2}\mathrm{tr}(\gamma(u,X^n(u),a)D^2\varphi(u,X^n(u)))\Big) + \mathbf{\Lambda_u}\nabla\varphi(u,X^n(u))^T\sigma(u,X^n(u),a)dW_u.\end{aligned}$$

Plug in the above result to (3.47), subtract $\varphi(t_n,x_n)$ from both sides and divide by $\frac{1}{n}$. We get

$$\begin{aligned}0 \leq E\Big[n\int_{t_n}^{\eta^n} &\mathbf{\Lambda_u}\Big(L(u,X^n(u),a)+\varphi_t(u,X^n(u))+\mu(u,X^n(u),a)\cdot\nabla\varphi(u,X^n(u))\\ &+\frac{1}{2}\mathrm{tr}(\gamma(u,X^n(u),a)D^2\varphi(u,X^n(u))) - r(u,X^n(u),a)\varphi(u,X^n(u))du\Big)\\ &+\int_{t_n}^{\eta^n}\mathbf{\Lambda_u}\nabla\varphi(u,X^n(u))^T\sigma(u,X^n(u),a)dW_u\Big] + e_n.\end{aligned}$$

We may assume without loss of generality that $\mathcal{O}$ is bounded, for otherwise we can take $\theta = \left(t + \frac{1}{n}\right) \wedge \theta_N$, where $\theta_N$ is the exit time of from the set $B(x_0, N) \cap \mathcal{O}$. Under the assumption that $\mathcal{O}$ is bounded, the stochastic integral term becomes a martingale. We note that for sufficiently large $n$, $\eta^n = t_n + \frac{1}{n}$. By our assumptions, passing to the limit as $n \to \infty$, we get

$$\begin{aligned} 0 \leq L(t_0, x_0, a) + \varphi_t(t_0, x_0, a) + \mu(t_0, x_0, a) \cdot \nabla\varphi(t_0, x_0) \\ + \frac{1}{2}\mathrm{tr}(\gamma(t_0, x_0, a) D^2\varphi(t_0, x_0)) - r(t_0, x_0, a)\varphi(t_0, x_0), \end{aligned}$$

since $ne_n \to 0$. Multiply by $-1$ both sides and take the supremum over all $a \in A$ to get

$$-\varphi_t(t_0, x_0) + H(t_0, x_0, \varphi(t_0, x_0), \nabla\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0.$$

For the second part of the theorem, let $\varphi$ be smooth and the point $(t_0, x_0) \in [0, T) \times \mathcal{O}$ satisfy

$$(v_* - \varphi)(t_0, x_0) = \min\left\{(v_* - \varphi)(t, x) : (t, x) \in [0, T] \times \overline{\mathcal{O}}\right\} = 0.$$

In view of the definition of the viscosity super-solution we need to show that

$$-\varphi_t(t_0, x_0) + H(t_0, x_0, \varphi(t_0, x_0), \nabla\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0.$$

Choose a sequence $(t_n, x_n) \to (t_0, x_0)$ as $n \to \infty$ such that $v(t_n, x_n) \to v_*(t_0, x_0) = \varphi(t_0, x_0)$ and

$$|v(t_n, x_n) - \varphi(t_n, x_n)| \leq |v(t_n, x_n) - v_*(t_0, x_0)| + |\varphi(t_0, x_0) - \varphi(t_n, x_n)| \leq \frac{1}{n^2}.$$

Set $\theta = t_n + \frac{1}{n}$ and choose $\alpha^n \in \mathcal{A}$ so that for

$$\eta^n := \left(t_n + \frac{1}{n}\right) \wedge \tau^{\alpha^n}_{t_n, x_n}$$

we have

$$v(t_n, x_n) \geq E\left[\int_{t_n}^{\eta^n} \mathbf{\Lambda_u} L(u, X^n(u), \alpha^n(u)) du + \mathbf{\Lambda_{\eta^n}} \varphi\left(\eta^n, X^n(\eta^n)\right)\right] - \frac{1}{n^2},$$

where $X^n = X^{\alpha^n}_{t_n, x_n}$. As before $v(t_n, x_n) = \varphi(t_n, x_n) + e_n$ and $e_n \leq \frac{1}{n^2}$. Again apply Ito to $\mathbf{\Lambda_u}\varphi(u, X^n(u))$, observe that the stochastic integral is a martingale, subtract $\varphi(t_n, x_n)$ from both sides and divide by $\frac{1}{n}$. We obtain

$$\begin{aligned} 0 \leq E\Bigg[n \int_{t_n}^{\eta^n} \mathbf{\Lambda_u}\Big(L(u, X^n(u), \alpha^n(u)) + \varphi_t(u, X^n(u)) + \mu(u, X^n(u), \alpha^n(u)) \cdot \nabla\varphi(u, X^n(u)) \\ + \frac{1}{2}\mathrm{tr}(\gamma(u, X^n(u), \alpha^n(u)) D^2\varphi(u, X^n(u))) - r(u, X^n(u), \alpha^n(u))\varphi(u, X^n(u)) du\Big) \\ + \int_{t_n}^{\eta^n} \mathbf{\Lambda_u} \nabla\varphi(u, X^n(u))^T \sigma(u, X^n(u), \alpha^n(u)) dW_u\Bigg] + e_n. \end{aligned}$$

By passing to the limit as $n \to \infty$

$$0 \leq \liminf_{n\to\infty} E\Bigg[n \int_{t_n}^{t_n+\frac{1}{n}} L(t_0, x_0, \alpha^n(u)) + \varphi_t(t_0, x_0) - r(t_0, x_0, \alpha^n(u))\varphi(t_0, x_0)$$
$$- \mu(t_0, x_0, \alpha^n(u)) \cdot \nabla\varphi(t_0, x_0) - \frac{1}{2}\mathrm{tr}(\gamma(t_0, x_0, \alpha^n(u))D^2\varphi(t_0, x_0))du\Bigg]$$

Observe that if we set

$$\hat{A} = \{(L(t_0, x_0, a), -r(t_0, x_0, a), \mu(t_0, x_0, a), \gamma(t_0, x_0, a)) : a \in A\}$$
$$H(t_0, x_0, v, p, \Gamma) = \sup_{a\in A}\Bigg\{ - L(t_0, x_0, a) + r(t_0, x_0, a)\varphi(t_0, x_0) - \mu(t_0, x_0, a) \cdot \nabla\varphi(t_0, x_0)$$
$$- \frac{1}{2}\mathrm{tr}(\gamma(t_0, x_0, a)D^2\varphi(t_0, x_0))\Bigg\},$$

then

$$H(t_0, x_0, v, p, \Gamma) = \sup\Bigg\{ - L(t_0, x_0, a) + r(t_0, x_0, a)\varphi(t_0, x_0) - \mu(t_0, x_0, a) \cdot \nabla\varphi(t_0, x_0)$$
$$- \frac{1}{2}\mathrm{tr}(\gamma(t_0, x_0, a)D^2\varphi(t_0, x_0)) : (L, -r, \mu, \gamma) \in \overline{\mathrm{co}(\hat{A})}\Bigg\}.$$

As in the determinisitic case, we conclude that

$$-\varphi_t(t_0, x_0) + H(t_0, x_0, \varphi(t_0, x_0), \nabla\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0.$$

□

### 3.3.4 Crandall-Ishii Lemma

#### Motivation

To motivate the Crandall-Ishii lemma, we return to our analysis of deterministic control problems. Suppose $u$ is continuous. Let $x \in \mathcal{O}$, where $\mathcal{O}$ is open. Define

$$D^+u(x) := \left\{\nabla\varphi(x) : \varphi \in C^1 \text{ and } (v - \varphi)(x) = \text{ local max } (v - \varphi)\right\},$$
$$D^-u(x) := \left\{\nabla\varphi(x) : \varphi \in C^1 \text{ and } (v - \varphi)(x) = \text{ local min } (v - \varphi)\right\}.$$

**Lemma 2.** (**L. Craig Evans**)
We have the following equivalences

$$D^+u(x) = \left\{p \in \mathbf{R}^d : \limsup_{y\to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0\right\},$$
$$D^-u(x) = \left\{p \in \mathbf{R}^d : \liminf_{y\to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0\right\}.$$

In view of the previous lemma, $u$ is a viscosity sub(super)-solution of

$$H(x, u(x), Du(x)) = 0$$

if and only if $H(x, u(x), p) \leq (\geq) 0$ for all $p \in D^+u(x)(D^-u(x))$ and for all $x \in \mathcal{O}$.

In deterministic control problems to prove the comparison argument, we introduced the following auxiliary function

$$\Phi(x,y) = u(x) - v(y) - \frac{1}{2\epsilon}|x-y|^2$$

for upper semicontinuous $u$ and lower semicontinuous $v$. Suppose that $\Phi(x,y)$ assumes its maximum at $(x_\epsilon, y_\epsilon)$. Then $x \mapsto u(x) - v(y_\epsilon) - \frac{1}{2\epsilon}|x-y_\epsilon|^2$ is maximized at $x_\epsilon$. Using the test function $\varphi(x) = v(y_\epsilon) + \frac{1}{2\epsilon}|x-y_\epsilon|^2$, we note that

$$\nabla\varphi(x_\epsilon) = \frac{x_\epsilon - y_\epsilon}{\epsilon} \text{ and } D^2\varphi(x_\epsilon) = \frac{I}{\epsilon}.$$

On the other hand, $y \mapsto v(y) - u(x_\epsilon) + \frac{1}{2\epsilon}|x-y_\epsilon|^2$ is minimized at $y_\epsilon$. Using $\varphi(y) = u(x_\epsilon) - \frac{1}{2\epsilon}|x_\epsilon - y|^2$ as a test function, we reach the conclusion

$$\nabla\varphi(y_\epsilon) = \frac{x_\epsilon - y_\epsilon}{\epsilon} \text{ and } D^2\varphi(y_\epsilon) = -\frac{I}{\epsilon}.$$

The classical maximum principle states that if $u-v$ assumes its maximum at $x$, then $\nabla u(x) = \nabla v(x)$ and $D^2u(x) \le D^2v(x)$. Although the condition $\nabla\varphi(x_\epsilon) = \nabla\varphi(y_\epsilon)$ resembles the first order condition of optimality, $D^2\varphi(x_\epsilon) > D^2\varphi(y_\epsilon)$ is not consistent with the classical maximum principle.

Next we return to stochastic control problems. As in the deterministic control problems we define

$$D^{+,2}u(x) := \left\{(\nabla\varphi(x), D^2\varphi(x)) : \varphi \in C^2 \text{ and } (v-\varphi)(x) = \text{ local max } (v-\varphi)\right\},$$
$$D^{-,2}u(x) := \left\{(\nabla\varphi(x), D^2\varphi(x)) : \varphi \in C^2 \text{ and } (v-\varphi)(x) = \text{ local min } (v-\varphi)\right\}.$$

**Lemma 3.** (**L. Craig Evans**)
We have the following equivalences

$$D^{+,2}u(x) = \left\{(p,\Gamma) \in \mathbf{R}^d \times \mathcal{S}_d : \limsup_{y\to x} \frac{u(y) - u(x) - p\cdot(y-x) - \frac{1}{2}\Gamma(y-x)\cdot(y-x)}{|y-x|} \le 0\right\},$$
$$D^{-,2}u(x) = \left\{(p,\Gamma) \in \mathbf{R}^d \times \mathcal{S}_d : \liminf_{y\to x} \frac{u(y) - u(x) - p\cdot(y-x) - \frac{1}{2}\Gamma(y-x)\cdot(y-x)}{|y-x|} \ge 0\right\}.$$

Also define

$$cD^{+,2}u(x) = \left\{(p,\Gamma) \in \mathbf{R}^d \times \mathcal{S}_d : \exists (x_n,p_n,\Gamma_n) \to (x,p,\Gamma) \text{ as } n\to\infty \text{ and } (p_n,\Gamma_n) \in D^{+,2}u(x_n)\right\},$$
$$cD^{-,2}u(x) = \left\{(p,\Gamma) \in \mathbf{R}^d \times \mathcal{S}_d : \exists (x_n,p_n,\Gamma_n) \to (x,p,\Gamma) \text{ as } n\to\infty \text{ and } (p_n,\Gamma_n) \in D^{-,2}u(x_n)\right\}.$$

**Theorem 27.** ***Characterization of viscosity solutions***
*$u$ is a viscosity sub(super)-solution of $u(x) + H(x,\nabla u(x), D^2u(x)) = 0$ on an open set $\mathcal{O} \subset \mathbf{R}^d$ if and only if*

$$u(x) + H(x,p,\Gamma) \le (\ge) 0 \quad \forall (p,\Gamma) \in cD^{+,2}u(x)\ (cD^{-,2}u(x)).$$

**Lemma 4. Crandall-Ishii Lemma**
Let $u, v : \mathcal{O} \subset \mathbf{R}^d \to \mathbf{R}$. Suppose $u$ is upper semi-continuous and $v$ lower semi-continuous and $\varphi \in C^2(\mathcal{O}\times\mathcal{O})$. Assume that

$$\Phi(x,y) = u(x) - v(y) - \varphi(x,y)$$

has its interior local maximum at $(x_0, y_0)$. Then for every $\lambda > 0$ there exists $p \in \mathbf{R}^d$ and $A_\lambda, B_\lambda \in \mathcal{S}_d$ such that

$$(p, A_\lambda) \in cD^{+,2}u(x_0), \quad (p, B_\lambda) \in cD^{-,2}v(y_0)$$

and

$$-\left(\frac{1}{\lambda} + \|D^2\varphi\|\right) I \leq \begin{pmatrix} A_\lambda & 0 \\ 0 & B_\lambda \end{pmatrix} \leq D^2\varphi + \lambda(D^2\varphi)^2. \tag{3.48}$$

*Remark.* We will use this lemma with $\varphi(x, y) = \frac{1}{2\epsilon}|x - y|^2$. In this case (3.48) becomes

$$-\frac{2}{\epsilon} I \leq \begin{pmatrix} A_\lambda & 0 \\ 0 & B_\lambda \end{pmatrix} \leq \frac{3}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{3.49}$$

if we choose $\lambda = \epsilon$.

*Remark.* If we have

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \leq c_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

for some $c_0 > 0$, then for any $\eta > 0$

$$(A-B)\eta\cdot\eta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \eta \\ \eta \end{pmatrix} \cdot \begin{pmatrix} \eta \\ \eta \end{pmatrix} \leq c_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \eta \\ \eta \end{pmatrix} \cdot \begin{pmatrix} \eta \\ \eta \end{pmatrix} \leq 0$$

showing that $A \leq B$.

*Remark.* If $u, v \in C^2(\mathcal{O})$, then by maximum principle

$$\nabla u(x_0) = \nabla v(y_0) = \frac{x_0 - y_0}{\epsilon} \text{ and } D^2\Phi(x_0, y_0) \leq 0.$$

This implies that

$$\begin{pmatrix} D^2u(x_0) & 0 \\ 0 & D^2v(y_0) \end{pmatrix} \leq \frac{1}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

The above calculations show that for smooth $u, v$ the Crandall-Ishii lemma is satisfied.

**Definition 8.** A function $\Psi : \mathcal{O} \to \mathbf{R}$ is semiconvex if $\Psi(x) + c_0|x|^2$ is convex for some $c_0 > 0$.

We state two properties of semiconvex functions without proof.

**Proposition 4. Semiconvex functions**

1. Let $u$ be a semiconvex function in $\mathcal{O}$ and $\hat{x}$ is an interior maximum, then $u$ is differentiable at $\hat{X}$ and $Du(\hat{x}) = 0$.

2. If $\Psi$ is semiconvex, then $D^2\Psi$ exists almost everywhere and $D^2\Psi \geq -c_0 I$, where $c_0$ is the semiconvexity constant.

Because smooth $u$ and $v$ satisfy the Crandall-Ishii lemma, one might expect to prove the lemma by first mollifying $u, v$ and then passing to the limit. However, it turns out that this approach does not work. In fact we need convex approximations. We will approximate $u$ by a semiconvex family $u^\lambda$ and $v$ by a semiconcave family $v^\lambda$. For this purpose, we introduce the sup-convolution.

**Definition 9.** Let $u : \mathcal{G} \subset \mathbf{R}^d \to \mathbf{R}$ be bounded and upper semicontinuos. Suppose $\mathcal{G}$ is closed and $\lambda > 0$. Define the sup-convolution $u^\lambda$ of $u$ by

$$u^\lambda(x) = \sup_{y \in \mathcal{G}} \left\{ u(x-y) - \frac{1}{2\lambda}|y|^2 \right\}.$$

**Proposition 5. Sup-convolution**

1. $u^\lambda(x)$ is semiconvex.
2. $u^\lambda(x) \downarrow u(x)$ as $\lambda \to 0$.
3. If $(p, \Gamma) \in D^{+,2}u^\lambda(x_0)$, then $(p, \Gamma) \in D^{+,2}u^\lambda(x_0 + \lambda p)$.
4. If $(0, \Gamma) \in cD^{+,2}u^\lambda(0)$, then $(0, \Gamma) \in cD^{+,2}u^\lambda(0)$.

**Proof.** The first claim follows easily noting that $\bar{u}^\lambda(x) = u^\lambda(x) + \frac{1}{2\lambda}|x|^2$ is convex, because for all $x, \eta$ we have

$$\bar{u}^\lambda(x) \leq \frac{1}{2}\bar{u}^\lambda(x+\eta) + \frac{1}{2}\bar{u}^\lambda(x-\eta).$$

For the second claim observe that $u^\lambda(x) \leq u^{\lambda'}(x)$ for $0 < \lambda < \lambda'$. Moreover, $u^\lambda(x) \geq u(x)$ so that $\liminf_{\lambda \to 0} u^\lambda(x) \geq u(x)$. Also for some $y_\lambda$,

$$u^\lambda(x) = u(x - y_\lambda) - \frac{1}{2\lambda}|y_\lambda|^2,$$

since $u$ is bounded. Hence,

$$\frac{1}{2\lambda}|y_\lambda|^2 \leq u(x - y_\lambda) - u(x) \;\Rightarrow\; |y_\lambda|^2 \leq 4\lambda\|u\|_\infty < \infty$$

so that $y_\lambda \to 0$. Therefore,

$$\limsup_{\lambda \to 0} u^\lambda(x) = \limsup_{\lambda \to 0} u(x - y_\lambda) - \frac{1}{2\lambda}|y_\lambda|^2 \leq u^*(x) = u(x).$$

To show the third claim, let $(p, \Gamma) \in D^{+,2}u^\lambda(x_0)$. Then we know that there exists smooth $\varphi$ such that $u^\lambda - \varphi$ attains its local maximum at $x_0$. Moreover, $D\varphi(x_0) = p$ and $D^2\varphi(x_0) = \Gamma$. Let $y_0$ be such that

$$u^\lambda(x_0) = u(y_0) - \frac{1}{2\lambda}|x_0 - y_0|^2.$$

Then for all $x$,

$$u(y_0) - \frac{1}{2\lambda}|x_0 - y_0|^2 - \varphi(x_0) \geq u(y_0) - \frac{1}{2\lambda}|x - y_0|^2 - \varphi(x)$$

so that $\alpha(x) = \varphi(x) + \frac{1}{2\lambda}|x - y_0|^2$ attains its minimum at $x_0$. The first order condition for $\alpha$ at $x_0$ gives that $y_0 = x_0 + \lambda p$. Because $x \mapsto u(x - y_0) - \frac{1}{2\lambda}|y_0|^2 - \varphi(x)$ has its max at $x_0$, then $(p, \Gamma) = (D\varphi(x_0), D^2\varphi(x_0)) \in D^{+,2}u(x_0 + \lambda p)$. The last claim is obtained by an approximation arguments and the previous parts. $\square$

**Theorem 28.** ***Aleksandrov's theorem***
*Let $\varphi : \mathbf{R}^d \to \mathbf{R}$ be semiconvex, i.e. $\varphi(x) + \frac{\lambda}{2}|x|^2$ is convex for $\lambda > 0$. Then $\varphi$ is twice differentiable Lebesgue almost everywhere.*

**Proof.** We will outline the main ideas, for a complete proof refer to [3]. Since $\varphi(x) + \frac{\lambda}{2}|x|^2$ is convex, it is locally Lipschitz, so by Rademacher's theorem it is almost everywhere differentiable. Moreover, $D^2\varphi + \lambda I$ exists as a positive distribution, hence it is a Radon measure, so $D^2\varphi$ exists almost everywhere and satsifies $D^2\varphi \geq -\lambda I$.

**Lemma 5. Jensen's lemma**
Let $\varphi : \mathbf{R}^d \to \mathbf{R}$ be semiconvex and $\hat{x}$ be a strict local maximum of $\varphi$. For $p \in \mathbf{R}^d$ set $\varphi_p(x) = \varphi(x) + p \cdot x$. For $r, \delta > 0$ define

$$K_{r,\delta} = \left\{x \in \overline{B}(\hat{x}, r) : \exists p \in \mathbf{R}^d, |p| < \delta \text{ and } \varphi_p \text{ has a maximum at x over } \overline{B}(\hat{x}, r)\right\}.$$

Then the Lebesgue measure of $K_{r,\delta}$ is positive, i.e. $\mathrm{Leb}(K_{r,\delta}) > 0$.

**Proof.** Choose $r$ so small enough that $\varphi$ has $\hat{x}$ as a unique maximum in $B(\hat{x}, r)$. Suppose for the moment that $\varphi$ is $C^2$. We claim that if $\delta_0$ is sufficiently small and $|p| < \delta_0$, then $x_p \in B(\hat{x}, r)$, where $x_p$ is the maximum of $\varphi_p$ with respect to $\overline{B}(\hat{x}, r)$. To prove this claim, let $x \in \partial B(\hat{x}, r)$. Since $\hat{x}$ is the unique strict maximizer, there exists $\mu_0 > 0$ such that

$$\varphi(\hat{x}) \geq \varphi(x) - \mu_0.$$

Then

$$\varphi_p(\hat{x}) - \varphi_p(x) \geq \varphi(\hat{x}) - \varphi(x) + p \cdot (\hat{x} - x) \geq \mu_0 - p|r|.$$

If we choose $|p| < \delta_0 := \frac{\mu_0}{2r}$, then

$$\varphi_p(x_p) - \varphi_p(x) \geq \mu_0 - |p|r > 0$$

to conclude $\varphi_p(x_p) > \sup_{x \in \partial\overline{B}(\hat{x},r)} \varphi_p(x)$. The next claim is $B(0, \delta) \subseteq D\varphi(K_{r,\delta})$ for $\delta \leq \delta_0$. So let $p \in B(0, \delta)$. As before, set $\varphi_p(x) = \varphi(x) + p \cdot x$. For $|p| \leq \delta < \delta_0$, $\varphi_p(x)$ assumes its maximum at $x_p$, where $x_p \in B(\hat{x}, r)$. Hence, $x_p \in K_{r,\delta}$. Because $x_p$ is an interior max, first order condition states that $p \in D\varphi(K_{r,\delta})$ proving the claim. Then for all $x \in K_{r,\delta}$, we obtain that $-\lambda I \leq D^2\varphi(x) \leq 0$, where $\lambda$ is the semiconvexity constant of $\varphi$. It follows that $|\det D^2\varphi(x)| \leq \lambda^d$ for all $x \in K_{r,\delta}$. Since $B(0, \delta) \subseteq D\varphi(K_{r,\delta})$,

$$c_d\delta^d = \mathrm{Leb}(B(0, \delta)) \leq \mathrm{Leb}(D\varphi(K_{r,\delta})) \leq \int_{K_{r,\delta}} |\det D^2\varphi(x)| dx \leq \mathrm{Leb}(K_{r,\delta})|\lambda|^d,$$

where $c_d$ is the volume of unit ball in $d$ dimensions. In view of these estimates for any $r \leq \bar{r}$ and $\delta \leq \bar{\delta} := \frac{\mu_0}{2\bar{r}}$

$$\mathrm{Leb}(K_{r,\delta}) \geq c_d \left(\frac{\delta}{\lambda}\right)^d.$$

In the case $\varphi$ is not smooth, approximate it by mollification with smooth functions $\varphi^n$ that have the same semi-convexity constant $\lambda$ and that converge uniformly to $\varphi$ on $\overline{B}(\hat{x}, r)$. If $n$ is sufficiently large, then the corresponding sets

obey the above estimates, i.e. $\mathrm{Leb}(K^n_{r,\delta}) > c_d \left(\frac{\delta}{\lambda}\right)^d$. Moreover, one can show that $K_{r,\delta} \supset \limsup_{n\to\infty} K^n_{r,\delta}$ to conclude

$$\mathrm{Leb}(K_{r,\delta}) \geq \limsup_{n\to\infty} \mathrm{Leb}(K^n_{r,\delta}) > 0.$$

□

**Lemma 6.** If for $f \in C(\mathbf{R}^d)$ and $Y \in \mathcal{S}_d$, $f(\xi) + \frac{\lambda}{2}|\xi|^2$ is convex and

$$\max_{\mathbf{R}^d} \left\{ f(\xi) - \frac{1}{2} \langle Y\xi, \xi \rangle \right\} = f(0)$$

then there exists $X \in \mathcal{S}_d$ such that $(0, X) \in cD^{+,2}f(0) \cap cD^{-,2}f(0)$ and $-\lambda I \leq X \leq Y$.

**Proof.** Clearly, $f(\xi) - \frac{1}{2}\langle Y\xi, \xi\rangle - |\xi|^4$ has a strict maximum at $\xi = 0$. By Aleksandrov's theorem and Jensen's lemma, for any $\delta > 0$,

$$K_{\delta,\delta} \cap \{x \in B(\hat{x}, \delta) : \varphi \text{ is twice differentiable at } x\} \neq \emptyset.$$

Hence, for all $\delta > 0$ sufficiently small there exists $p_\delta$ such that $|p_\delta| \leq \delta$ and $f(\xi) + p_\delta \cdot \xi - \frac{1}{2}\langle Y\xi, \xi\rangle - |\xi|^4$ has a maximum at $\xi_\delta$ with $|\xi_\delta| \leq \delta$. Also, $f$ is twice differentiable at $\xi_\delta$. Since $|p_\delta|, |\xi_\delta| \leq \delta$ and by semiconvexity we have

$$Df(\xi_\delta) = O(\delta), \quad -\lambda I \leq D^2 f(\xi_\delta) \leq Y + O(\delta^2).$$

Moreover, $(Df(\xi_\delta), D^2 f(\xi_\delta)) \in D^{2,+}f(\xi_\delta) \cap D^{2,-}f(\xi_\delta)$. Since $D^2 f(\xi_\delta)$ is uniformly bounded, it has a convergent subsequence to some $X \in \mathcal{S}_d$ and by Bolzano-Weierstrass, $\xi_\delta$ tends to 0 by passing to a subsequence if necessary. Therefore, we conclude that $(0, X) \in D^{2,+}f(0) \cap D^{2,-}f(0)$ and $-\lambda I \leq X \leq Y$. □

**Proof. Proof of Crandall-Ishii Lemma**
We will prove the Crandall-Ishii Lemma for $\varphi(x, y) = \frac{1}{2\epsilon}|x - y|^2$. Without loss of generality, we can take $\mathcal{O} = \mathbf{R}^d$, $x_0 = y_0 = 0$ and $u(0) = v(0) = 0$. For the proof of these simplifications refer to [3]. To prove the lemma, we need to show that there exists $A, B \in \mathcal{S}_d$ such that

$$(0, A) \in cD^{+,2}u(0), \quad (0, B) \in cD^{-,2}v(0)$$

and

$$-\frac{2}{\epsilon} I \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq \frac{3}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

Because of the simplifications

$$\Phi(x, y) = u(x) - v(y) - \frac{1}{2\epsilon}|x - y|^2 \leq 0, \forall (x, y) \in \mathcal{O} \times \mathcal{O},$$

or for

$$K = \begin{pmatrix} \frac{I}{\epsilon} & -\frac{I}{\epsilon} \\ -\frac{I}{\epsilon} & \frac{I}{\epsilon} \end{pmatrix},$$

$$u(x) - v(y) \leq \frac{1}{2} K \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Taking the sup-convolution of both sides

$$u^\lambda(x) - v^\lambda(y) = (u(x) - v(y))^\lambda \leq \left[\frac{1}{2}K\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right]^\lambda$$

By a direct computation

$$\left[\frac{1}{2}K\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right]^\lambda = K(I + \gamma K)\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

for every $\gamma > 0$ with $\frac{1}{\lambda} = \frac{1}{\gamma} + \|K\|$. We choose $\gamma = \frac{1}{\epsilon}$ and since $\|K\| = \frac{1}{\epsilon}$, we get $\lambda = \frac{2}{\epsilon}$. Moreover, it can be shown that $u^\lambda(0) = v^\lambda(0) = 0$ and

$$u(x) - v(y) \leq [K(I + \epsilon K)]\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can apply Lemma (6) to get the existence $A, B \in \mathcal{S}_d$ such that

$$(0, A) \in cD^{+,2}u^\lambda(0), \quad (0, B) \in cD^{-,2}v^\lambda(0)$$

and

$$-\frac{2}{\epsilon}I \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq K + \epsilon K^2 = \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

But we know that $(0, A) \in cD^{+,2}u^\lambda(0) = cD^{+,2}u(0)$ and $(0, B) \in cD^{-,2}v^\lambda(0) = cD^{-,2}v(0)$ to conclude the proof. □

**Theorem 29.** ***Comparison Theorem***
*Let $\mathcal{O} \subseteq \mathbf{R}^d$ be an open and bounded set. Suppose $u$ is a viscosity subsolution and $v$ is a viscosity supersolution of*

$$0 = u(x) + H(x, \nabla v(x), D^2 v(x)) \quad \forall x \in \mathcal{O}$$
$$H(x, p, \Gamma) = \sup_{a \in A}\left\{-L(t, x, a) - \mu(t, x, a) \cdot p - \frac{1}{2}tr(\gamma(t, x, a)\Gamma)\right\}.$$

*If $u^*(x) \leq v_*(x)$ for $x \in \partial\mathcal{O}$, then $u^*(x) \leq v_*(x)$ for all $x \in \overline{\mathcal{O}}$.*

**Proof.** As usual consider the auxiliary function

$$\Phi(x, y) = u^*(x) - v_*(y) - \frac{1}{2\epsilon}|x - y|^2$$

which is maximized at $(x_\epsilon, y_\epsilon) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$. If $x_\epsilon$ or $y_\epsilon \in \partial\mathcal{O}$ or on a subsequence, the claim follows easily, because $x_\epsilon, y_\epsilon$ both tend to an element $x \in \partial\mathcal{O}$. So consider $x_\epsilon, y_\epsilon \in \mathcal{O}$. By Crandall-Ishii Lemma there exist $A_\epsilon, B_\epsilon \in \mathcal{S}_d$ such that for $p_\epsilon = \frac{x_\epsilon - y_\epsilon}{\epsilon}$

$$(p_\epsilon, A_\epsilon) \in cD^{+,2}u^*(x_\epsilon), \quad (p_\epsilon, B_\epsilon) \in cD^{-,2}v_*(y_\epsilon)$$

and

$$-\frac{2}{\epsilon}I \leq \begin{pmatrix} A_\epsilon & 0 \\ 0 & -B_\epsilon \end{pmatrix} \leq \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Since $(p_\epsilon, A_\epsilon) \in cD^{+,2}u^*(x_\epsilon)$ and $(p_\epsilon, B_\epsilon) \in cD^{-,2}v_*(y_\epsilon)$,

$$\begin{aligned} u^*(x_\epsilon) + H(x_\epsilon, p_\epsilon, A_\epsilon) &\leq 0 \\ v_*(y_\epsilon) + H(y_\epsilon, p_\epsilon, B_\epsilon) &\geq 0. \end{aligned}$$

Subtract the second equation from the first, note the form of the Hamiltonian to get

$$\begin{aligned} u^*(x_\epsilon) - v_*(y_\epsilon) &\leq \sup_{a\in A}\Big\{|L(x_\epsilon, a) - L(y_\epsilon, a)| + |\mu(x_\epsilon, a) - \mu(y_\epsilon, a)||p_\epsilon| \\ &\quad + \frac{1}{2}\mathrm{tr}(\gamma(x_\epsilon, a)A_\epsilon - \gamma(y_\epsilon, a)B_\epsilon)\Big\} \\ &\leq K_L|x_\epsilon - y_\epsilon| + K_\mu\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} + \sup_{a\in A}\left\{\frac{1}{2}\mathrm{tr}(\gamma(x_\epsilon, a)A_\epsilon - \gamma(y_\epsilon, a)B_\epsilon)\right\}, \end{aligned}$$

since $L$ and $\mu$ are uniformly Lipschitz. As it is standard in comparison arguments, $|x_\epsilon - y_\epsilon|$ and $\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon}$ tend to zero as $\epsilon \to 0$. However, the last term tends to zero also, because

$$\begin{aligned} &\begin{pmatrix} A_\epsilon & 0 \\ 0 & -B_\epsilon \end{pmatrix}\begin{pmatrix} \sigma(x_\epsilon, a) \\ \sigma(y_\epsilon, a) \end{pmatrix} \cdot \begin{pmatrix} \sigma(x_\epsilon, a) \\ \sigma(y_\epsilon, a) \end{pmatrix} \\ &\leq \frac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix} \sigma(x_\epsilon, a) \\ \sigma(y_\epsilon, a) \end{pmatrix} \cdot \begin{pmatrix} \sigma(x_\epsilon, a) \\ \sigma(y_\epsilon, a) \end{pmatrix} \\ &= \frac{3}{\epsilon}|\sigma(x^\epsilon, a) - \sigma(y^\epsilon, a)| \leq \frac{3}{\epsilon}K_\sigma|x_\epsilon - y_\epsilon|^2. \end{aligned}$$

□

# Chapter 4

# Some Markov Chain Examples

## 4.1 Multiclass Queueing Systems

Martins, Shreve and Soner study a family of 2-station queueing networks in [12]. In the $n$th network of this family, two types of customers arrive at station 1 exponentially with rates $\lambda_1^{(n)}, \lambda_2^{(n)}$ respectively and they are served exponentially with respective rates $\mu_1^{(n)}, \mu_2^{(n)}$. After service, type 1 customers leave the system, whereas type 2 customers proceed to station 2, where they are redesignated as type 3 customers and get served exponentially with rate $\mu_3^{(n)}$. The controls are $\{(Y(t), U(t)), 0 \leq t < \infty\}$, where $Y(t)$ and $U(t)$ are left-continuous $\{0, 1\}$ valued processes. $Y(t)$ indicates whether station 1 is active ($Y(t) = 1$) or idle ($Y(t) = 0$). $U(t) = 1$ represents that station 1 is serving type 1 customer and $U(t)$ is 0 if type 2 customer gets serviced by station 1. Let $Q_i^{(n)}$ be the number of class $i$ customers queued or undergoing service at time $t$ and $Q^{(n)}(t) = (Q_1^{(n)}(t), Q_2^{(n)}(t), Q_3^{(n)}(t))$ denotes the vector of the queue length at time $t$. The vector of scaled queue length process is

$$Z^{(n)}(t) = \frac{1}{\sqrt{n}} Q^{(n)}(nt).$$

For fixed controls $(y, u) \in \{0, 1\}^2$, this is a Markov chain with lattice space $L^{(n)} = \left\{ \frac{k}{\sqrt{n}} : k = 0, 1, \ldots \right\}^3$. The infinitesimal generator of this Markov chain is given as

$$\begin{aligned}
(\mathcal{L}^{n,y,u}\varphi)(z) = {} & n\lambda_1^{(n)} \left[ \varphi\left( z + \frac{1}{\sqrt{n}} e_1 \right) - \varphi(z) \right] + n\lambda_2^{(n)} \left[ \varphi\left( z + \frac{1}{\sqrt{n}} e_2 \right) - \varphi(z) \right] \\
& + n\mu_1^{(n)} yu \left[ \varphi\left( z - \frac{1}{\sqrt{n}} e_1 \right) - \varphi(z) \right] \mathbf{1}_{\{z_1 > 0\}} \\
& + n\mu_2^{(n)} y(1-u) \left[ \varphi\left( z - \frac{1}{\sqrt{n}} e_2 + \frac{1}{\sqrt{n}} e_3 \right) - \varphi(z) \right] \mathbf{1}_{\{z_2 > 0\}} \\
& + n\mu_3^{(n)} \left[ \varphi\left( z - \frac{1}{\sqrt{n}} e_3 \right) - \varphi(z) \right] \mathbf{1}_{\{z_3 > 0\}},
\end{aligned}$$

where $z = (z_1, z_2, z_3)$ and $e_i$ is the $i$th unit vector. Define the holding cost function $h(z) = \sum_{i=1}^{3} c_i z_i$, where $c_i > 0$ are the costs per unit time of holding one class $i$ customer. Given an initial condition $Z^{(n)}(0) = z \in L^{(n)}$ and controls $(Y(\cdot), U(\cdot))$ define the cost functional for some constant $\alpha > 0$

$$J^{(n)}_{Y,U}(z) = E\left\{\int_0^\infty e^{-\alpha t} h(Z^{(n)}(t))dt\right\} = \frac{1}{n^{3/2}} E\left\{\int_0^\infty e^{-\alpha t/n} h(Q^{(n)}(t))dt\right\}$$

and the value function as

$$J^{(n)}_*(z) = \inf_{Y,U} J^{(n)}_{Y,U}(z).$$

For $\varphi : L^{(n)} \to \mathbf{R}$, define $\mathcal{L}^{(n),*}$ acting on $\varphi$ by

$$\mathcal{L}^{(n),*}\varphi(z) = \min\left\{\mathcal{L}^{n,y,u}\varphi(z) : (y,u) \in \{0,1\}^2\right\} \quad \forall z \in L^{(n)}.$$

Then the value function $J^{(n)}_*$ is the unique solution of the HJB equation

$$\alpha\varphi(z) - \mathcal{L}^{(n),*}\varphi(z) - h(z) = 0 \quad z \in L^{(n)}.$$

Moreover, we can also express $\mathcal{L}^{(n),*}(z) = \min\left\{\mathcal{L}^{n,y,u} : (y,u) \in [0,1]^2\right\}$. We also impose the heavy traffic assumption

$$\frac{\lambda^{(n)}_1}{\mu^{(n)}_1} + \frac{\lambda^{(n)}_2}{\mu^{(n)}_2} = 1 - \frac{b^{(n)}_1}{\sqrt{n}} \quad \frac{\lambda^{(n)}_2}{\mu^{(n)}_3} = 1 - \frac{b^{(n)}_2}{\sqrt{n}},$$

where $\lambda_j = \lim_{n\to\infty} \lambda^{(n)}_j$, $\mu_i = \lim_{n\to\infty} \mu^{(n)}_i$ and $b_j = \lim_{n\to\infty} b^{(n)}_j$ are defined for $j = 1, 2$ and $i = 1, 2, 3$ and positive. Furthermore, they satisfy

$$\sup_n \left[\sqrt{n}\sum_{j=1}^{2} |\lambda^{(n)}_j - \lambda_j| + \sqrt{n}\sum_{i=1}^{3} |\mu^{(n)}_i - \mu_i| + \sum_{j=1}^{2} |b^{(n)}_j - b_j|\right] < \infty.$$

We are interested in the heavy traffic limit of the value function. Therefore, we need an upper bound independent of $n$ for the non-negative functions $\left\{J^{(n)}_*\right\}_{n=1}^{\infty}$. So we consider

$$\begin{aligned}
(\mathcal{L}^{n,y,u}\varphi)(z) = {} & n\lambda^{(n)}_1\left[\varphi_1(z)\frac{1}{\sqrt{n}} + \frac{1}{2}\varphi_{11}(z)\frac{1}{n}\right] + n\lambda^{(n)}_2\left[\varphi_2(z)\frac{1}{\sqrt{n}} + \frac{1}{2}\varphi_{22}(z)\frac{1}{n}\right] \\
& + n\mu^{(n)}_1 yu\left[-\varphi_1(z)\frac{1}{\sqrt{n}} + \frac{1}{2}\varphi_{11}(z)\frac{1}{n}\right] \\
& + n\mu^{(n)}_2 y(1-u)\left[-\varphi_2(z)\frac{1}{\sqrt{n}} + \varphi_3(z)\frac{1}{\sqrt{n}} + \frac{1}{2}\varphi_{22}(z)\frac{1}{n} + \frac{1}{2}\varphi_{33}(z)\frac{1}{n} - \varphi_{23}(z)\frac{1}{n}\right] \\
& + n\mu^{(n)}_3\left[-\varphi_3(z)\frac{1}{\sqrt{n}} + \frac{1}{2}\varphi_{33}(z)\frac{1}{n}\right] + O\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

where the subscripts denote the derivatives of $\varphi$, for instance $\varphi_i$ represents the derivative with respect to $i$th coordinate. In the asymptotic expansion above, if we choose $y = 1$ and $u = \frac{\lambda_1}{\mu_1}$ we find that $(\mathcal{L}^{n,y,u}\varphi)(z) \approx O(1)$ so that we can pass to the limit of the value function as $n \to \infty$.

## 4.2 Production Planning

Fleming, Sethi and Soner, [7], consider an infinite horizon stochastic production planning problem. The demand rate $z(t)$ is a continuous-time Markov chain with a finite state space $Z$. The state $y(t)$ is the inventory level taking values in $\mathbf{R}^d$. The production rate $p(t)$ is the control valued in a closed, convex subset $K$ of $\mathbf{R}^N$. The generator $\mathcal{L}$ of the Markov chain $z(t)$ is given by

$$\mathcal{L}\varphi(z) = \sum_{z' \neq z} q_{zz'}[\varphi(z') - \varphi(z)],$$

where $q_{zz'}$ is the jumping rate from state $z$ to $z'$. The inventory level $y(t)$ evolves as

$$dy(t) = [B(z(t))p(t) + c(z(t))]dt, \quad t \geq 0.$$

The control process $P = \{p(t,\omega) : t \geq 0, \omega \in \Omega\}$ is admissible, i.e. $P \in \mathcal{A}$, if $P$ is adapted to the filtration generated by $z(t)$, $p(t,\omega) \in K$ for all $t \geq 0, \omega \in \Omega$ and

$$\sup\{|p(t,\omega)| : t \geq 0, \omega \in \Omega\} < \infty.$$

For every $P \in \mathcal{A}$, $y(0) = y$ and $z(0) = z$ let

$$J(y,z,P) = E\int_0^\infty e^{-\alpha t} l(y(t), p(t))dt,$$

where we assume that $l$ is convex on $\mathbf{R}^d \times K$. We also impose appropriate growth conditions and a lower bound on $l$. Then we express the value function as

$$v(y,z) = \inf\{J(y,z,P) : P \in \mathcal{A}\}.$$

We make two claims without proof. For any $z \in Z$, $v(\cdot, z)$ is convex on $\mathbf{R}^d$. The associated dynamic programming equation with the control problem is

$$\alpha v(y,z) - \inf_{p \in K}\{l(y,p) + [B(z)p + c(z)] \cdot \nabla v(y,z)\} - [\mathcal{L}v(y,\cdot)](z) = 0. \quad (4.1)$$

*Remark.*

1. $v$ is differentiable in the $y$-direction at $(y,z)$ if and only if $D_y^+ v(y,z)$ and $D_y^- v(y,z)$ consists only of the gradient $\nabla v(y,z)$.
2. If $v$ is convex in $y$, then $D_y^+ v(y,z)$ is empty unless $v$ is differentiable there.
3. $D_y^- v(y,z) = \overline{\text{co}}\Gamma(y,z)$, where

   $$\Gamma(y,z) = \left\{ r = \lim_{n\to\infty} \nabla v(y_n, z) : y_n \to y \text{ as } n \to \infty \text{ and } v(\cdot, z) \text{ is differentiable at } y_n \right\},$$

   and $\overline{\text{co}}$ denotes the convex closure.

We make the further assumption on

$$H(y,z,r) = \inf_{p \in K}[l(y,p) + (B(z)p + c(z)) \cdot r].$$

If for $y, z$ and $r_1, r_2$, $H(y,z,\lambda r_1 + (1-\lambda)r_2) = c$ for all $0 \leq \lambda \leq 1$, then $r_1 = r_2$.

**Theorem 30.** *Under the above assumption if $v$ is a viscosity solution of the dynamic programming equation (4.1) and $v(\cdot, z)$ is convex for all $z$, then $\nabla v(y,z)$ exists for all $(y,z)$ and $\nabla v(\cdot, z)$ is continuous on $\mathbf{R}^d$.*

**Proof.** According to the above remark it suffices to show that $D_y^- v(y,z)$ consists of a singleton. If $v(\cdot,z)$ is differentiable at $y_n$, then

$$\alpha v(y_n,z) - H(y_n,z,\nabla v(y_n,z)) - \mathcal{L}v(y_n,z) = 0.$$

By definition of $\Gamma(y,z)$, if we pass to the limit $y_n \to y$ as $n \to \infty$ we obtain for every $r \in \Gamma(y,z)$

$$\alpha v(y,z) - H(y,z,r) - \mathcal{L}v(y,z) = 0.$$

For fixed $(y,z)$ and for every $r \in \Gamma(y,z)$

$$H(y,z,r) = \alpha v(y,z) - \mathcal{L}v(y,z) = c_0.$$

Because $H(y,z,\cdot)$ is concave, for any $q \in \overline{\text{co}}\Gamma(y,z)$, we get that $H(y,z,q) \geq 0$. On the other hand, we know that $D_y^- v(y,z) = \overline{\text{co}}\Gamma(y,z)$. So it follows that since $v$ is a viscosity solution, for $q \in \overline{\text{co}}\Gamma(y,z)$

$$H(y,z,q) \leq \alpha v(y,z) - \mathcal{L}v(y,z) = c_0.$$

Thus, for fixed $(y,z)$ $H(y,z,q) = c_0$ on the convex set $D_y^- v(y,z)$. By the assumption on $H$, $D_y^- v(y,z)$ is a singleton. □

Next we take $K = [0,\infty)$, $d = 1$ and $Z = \{z_1,\ldots,z_M\}$. Moreover, the inventory level evolves as

$$dy(t) = [p(t) - z(t)]dt.$$

For a strictly convex $C^2$ function $c$ on $[0,\infty)$ with $c(0) = c'(0)$ and a non-negative convex function $h$ satisfying $h(0) = 0$ we set $l(y,p) = c(p) + h(y)$. We seek to minimize

$$J(y,z,p) = E\int_0^\infty e^{-\alpha t}[h(y(t)) + c(p(t))]dt.$$

The corresponding Hamiltonian is given by $H(y,z,r) = F(r) - zr + h(y)$, where $F(r) = \min_{p\geq 0}[pr + c(p)]$. The assumption we placed on the general Hamiltonian is satisfied, since $z > 0$ for all $z \in Z$ and $F(r)$ is strictly concave for $r < 0$ and $F(r) = 0$ for $r \geq 0$. Then, $v(y,z)$ is the unique viscosity solution of the corresponding dynamic programming equation. Moreover, the optimal feedback production policy is given by

$$p^*(y,z) = \begin{cases} (c')^{-1}(-v_y(y,z)) & \text{if } v_y(y,z) < 0 \\ 0 & \text{if } v_y(y,z) \geq 0 \end{cases}$$

Since $v$ is convex in $y$, $(c')^{-1}$ is increasing, we conclude that $p^*$ is non-increasing in $y$. Therefore, the ordinary differential equation

$$dy(t) = [p^*(y(t),z(t)) - z(t)]dt$$

has a unique solution. By a verification argument we conclude that $p^*$ is the optimal feedback policy.

# Chapter 5

# Appendix

## 5.1 Equivalent Definitions of Viscosity Solutions

For simplicity consider the first order nonlinear equation

$$\begin{aligned} 0 &= H(x, u(x), \nabla u(x)) \quad \text{where,} \\ H &: (x, u, p) \in \mathcal{O} \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}, \end{aligned} \tag{5.1}$$

where $\mathcal{O}$ is an open set in $\mathbf{R}^d$. One could formulate equivalent characterizations for second order parabolic Hamiltonians by replacing $C^1$ conditions with $C^2$. We assume that $H$ is continuous for all components.

**Theorem 31.** *The following definitions are equivalent:*

1. *$u \in L^\infty_{loc}(\mathcal{O})$ is a viscosity subsolution of (5.1) if*

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \varphi)(x)$$

*for some $\varphi \in C^\infty(\mathbf{R}^d)$ and $x_0 \in \mathcal{O}$, then*

$$H(x_0, u^*(x_0), D\varphi(x_0)) \leq 0.$$

2. *$u \in L^\infty_{loc}(\mathcal{O})$ is a viscosity subsolution of (5.1) if*

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \varphi)(x) = 0$$

*for some $\varphi \in C^\infty(\mathbf{R}^d)$ and $x_0 \in \mathcal{O}$, then*

$$H(x_0, \varphi(x_0), D\varphi(x_0)) \leq 0.$$

3. *$u \in L^\infty_{loc}(\mathcal{O})$ is a viscosity subsolution of (5.1) if*

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{N}}}(u^* - \varphi)(x) = 0$$

*for some $\varphi \in C^\infty(\mathbf{R}^d)$, $\mathcal{N} \subset \mathcal{O}$ open and bounded and $x_0 \in \mathcal{N}$, then*

$$H(x_0, \varphi(x_0), D\varphi(x_0)) \leq 0.$$

4. *$u \in L^\infty_{loc}(\mathcal{O})$ is a viscosity subsolution of (5.1) if*

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \varphi)(x)$$

*for some $\varphi \in C^\infty(\mathbf{R}^d)$ and strict maximizer $x_0 \in \mathcal{O}$, then*

$$H(x_0, u^*(x_0), D\varphi(x_0)) \leq 0.$$

5. *$u \in L^\infty_{loc}(\mathcal{O})$ is a viscosity subsolution of (5.1) if*

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{N}}}(u^* - \varphi)(x) = 0$$

*for some $\varphi \in C^1(\mathbf{R}^d)$, $\mathcal{N} \subset \mathcal{O}$ open and bounded and $x_0 \in \mathcal{N}$, then*

$$H(x_0, \varphi(x_0), D\varphi(x_0)) \leq 0.$$

**Proof.** It is straightforward to see (1) $\Rightarrow$ (2). For (2) $\Rightarrow$ (1). Assume that for $\varphi \in C^\infty$ and $x_0 \in \mathcal{O}$ we have

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \varphi)(x)$$

Then set $\widehat{\varphi}(x) = \varphi(x) + u^*(x_0) - \varphi(x_0)$. Clearly,

$$(u^* - \widehat{\varphi})(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \widehat{\varphi})(x) = 0$$

so that

$$H(x_0, u^*(x_0), \nabla\widehat{\varphi}(x_0)) \leq 0.$$

But $\nabla\widetilde{\varphi}(x_0) = \nabla\varphi(x_0)$ so that (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (1) follows immediately. For the converse statement, assume that for $\varphi \in C^\infty$, $\mathcal{N} \subset \mathcal{O}$ open, bounded and $x_0 \in \mathcal{N}$ we have

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{N}}}(u^* - \varphi)(x) = 0$$

Our aim is to construct $\widehat{\varphi} \in C^\infty$ such that

$$(u^* - \widehat{\varphi})(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \widehat{\varphi})(x)$$

and $\varphi(x) = \widehat{\varphi}(x)$ for all $x \in B(x_0, \epsilon)$ for some $\epsilon > 0$. Choose $\epsilon > 0$ so that $B(x_0, 2\epsilon) \subset \mathcal{N}$. Define for $k = 1, 2, \ldots$

$$m_k = \sup_{B(x_0, k\epsilon)} u^*(x).$$

Since $u^*$ is locally bounded, we have $m_k < \infty$ and $m_k$ is non-decreasing. We will use the fact that there exists a function $\eta \in C^\infty$ such that $\eta(x) = 0$ for $x \leq \frac{1}{3}$ and $\eta(x) = 1$ for $x \geq \frac{2}{3}$. Then consider

$$\Psi(x) = m_{\lceil \frac{|x|}{\epsilon} \rceil} + \left(m_{\lceil \frac{|x|}{\epsilon} \rceil + 1} - m_{\lceil \frac{|x|}{\epsilon} \rceil}\right) \eta\left(\frac{|x|}{\epsilon} - \left\lfloor \frac{|x|}{\epsilon} \right\rfloor\right).$$

The function $\Psi \in C^\infty$ and $\Psi(x) \geq u^*(x)$ for all $x$. Morevover, there exists another smooth function $\chi \in C^\infty$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ for $x \in B(x_0, \epsilon)$ and $\chi \equiv 0$ for $x \notin B(x_0, 2\epsilon)$. Set

$$\widehat{\varphi}(x) = \chi(x)\varphi(x) + (1 - \chi(x))\Psi(x).$$

Since $u^*(x) \leq \varphi(x)$ and $u^*(x) \leq \Psi(x)$, we have $u^*(x) \leq \widehat{\varphi}(x)$. Hence, (1) $\Rightarrow$ (3).

The implication (1) $\Rightarrow$ (4) follows immediately. For the converse, if there exists $\varphi \in C^\infty$, $x_0 \in \mathcal{O}$ such that

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{O}}}(u^* - \varphi)(x)$$

then construct

$$\widehat{\varphi}(x) = \varphi(x) + |x - x_0|^2.$$

Clearly,

$$(u^* - \widehat{\varphi})(x_0) = (u^* - \varphi)(x_0) \geq (u^* - \varphi)(x) > (u^* - \widehat{\varphi})(x)$$

for all $x \neq x_0$. Moreover, $\nabla\widehat{\varphi}(x_0) = \nabla\varphi(x_0)$. This concludes the implication (4) $\Rightarrow$ (1).

Next we show that (5) and (3) are equivalent. (5) $\Rightarrow$ (3) is trivial. Conversely, by similar arguments as above we can consider without loss of generality $\varphi \in C^1$, $\mathcal{N} \subset \mathcal{O}$ open, bounded and $x_0 \in \mathcal{N}$ a strict maximizer of

$$(u^* - \varphi)(x_0) = \max_{\overline{\mathcal{N}}}(u^* - \varphi)(x) = 0.$$

Consider the function $\eta \in C^\infty$ with the properties $0 \leq \eta \leq 1$, $\int \eta(x)dx = 1$ and $\text{supp}(\eta) \subset B(0,1)$. Take $\varphi^\epsilon(x) = \eta^\epsilon * \varphi(x)$, where $\eta^\epsilon(x) = \frac{1}{\epsilon^d}\eta\left(\frac{|x|}{\epsilon}\right)$. Then $\varphi^\epsilon \in C^\infty$ and $\varphi^\epsilon \to \varphi$ as $\epsilon \to 0$ uniformly on $\overline{\mathcal{N}}$. Let $x^\epsilon \in \overline{\mathcal{N}} \cap \overline{\mathcal{O}}$ be the maximizer of

$$\max_{\overline{\mathcal{N}} \cap \overline{\mathcal{O}}} u^*(x) - \varphi^\epsilon(x) = u^*(x^\epsilon) - \varphi^\epsilon(x^\epsilon). \tag{5.2}$$

Since $\{x^\epsilon\}$ belongs to the compact set $\overline{\mathcal{N}} \cap \overline{\mathcal{O}}$, by passing to a subsequence if necessary, $x^\epsilon \to \bar{x}$. We now claim that $x^\epsilon \to x_0$ and $u^*(x^\epsilon) \to u^*(x_0)$ using $x_0$ is a strict maximizer.

$$\begin{aligned} 0 = u^*(x_0) - \varphi(x_0) &\leq \max_{\overline{\mathcal{N}} \cap \overline{\mathcal{O}}} u^*(x) - \varphi^\epsilon(x) \\ &\leq u^*(x^\epsilon) - \varphi^\epsilon(x^\epsilon). \end{aligned}$$

This is true for any $\epsilon$ so that by upper-semicontinuity of $u^*$ we get

$$\begin{aligned} 0 = u^*(x_0) - \varphi(x_0) &\leq \lim_{\epsilon \downarrow 0} u^*(x^\epsilon) - \varphi^\epsilon(x^\epsilon) \\ &\leq u^*(\bar{x}) - \varphi(\bar{x}) \leq u^*(x_0) - \varphi(x_0). \end{aligned}$$

This establishes that $x^\epsilon \to x_0$ and $u^*(x^\epsilon) \to u^*(x_0)$. Because (5.2), by definition of viscosity subsolution we obtain

$$H(x^\epsilon, u^*(x^\epsilon), \varphi^\epsilon(x^\epsilon)) \leq 0.$$

Since by assumption $H$ is continuous in all of its components, we retrieve

$$H(x_0, u^*(x_0), \varphi(x_0)) \leq 0.$$

□

*Remark.* One could easily prove more equivalent versions of the above statements by just imitating the proofs. For instance, all formulations have a strict maximum version.

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