$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad f \in C^2(\Omega)$$

The Hessian Matrix $\overline{=} H_f(x_0) = \left(\frac{\partial f}{\partial x^i \partial x^j}\right)$

$$= \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \cdots & \frac{\partial^2 f}{\partial x^1 \partial x^n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x^n \partial x^1} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{pmatrix}$$

Thm (2n order Taylor formula) for $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)\, H_f(x_0)(x - x_0)^t$$

$$+ R_2(x; x_0)$$

where $$\lim_{x \to x_0} \frac{R(x, x_0)}{|x - x_0|^2} = 0$$

Extrema of functions $f : \mathbb{R}^n \to \mathbb{R}$.

Thm If f is differentiable and $x_0 \in \mathbb{R}^n$ is a local extremum then $\nabla f(x_0) = 0$.

Defn A stationary (or critical) point of f is a point x_0 where $\nabla f(x_0) = 0$.

Thm Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $f \in C^2$. Let x_0 be a critical point of f (ie $\nabla f(x_0) = 0$) with $\det(\mathrm{Hess}_f(x_0)) \neq 0$

1) If $\mathrm{Hess}_f(x_0) > 0$ then a is a local min.
2) If $\mathrm{Hess}_f(x_0) < 0$ then a is a local max.
3) Otherwise it is a saddle point (ie if $\mathrm{Hess}_f(x_0)$ is indefinite)

Cor $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$, $a = (x_0, y_0)$ a critical point, $f \in C^2$

1) if $\frac{\partial^2 f}{\partial x^2}(a) < 0$ and $\det(\mathrm{Hess}_f(a)) > 0$ then a is a local max of f.

2) if $\frac{\partial^2 f}{\partial x^2}(a) > 0$, and $\det(\mathrm{Hess}_f(a)) > 0$ then a is a local min of f

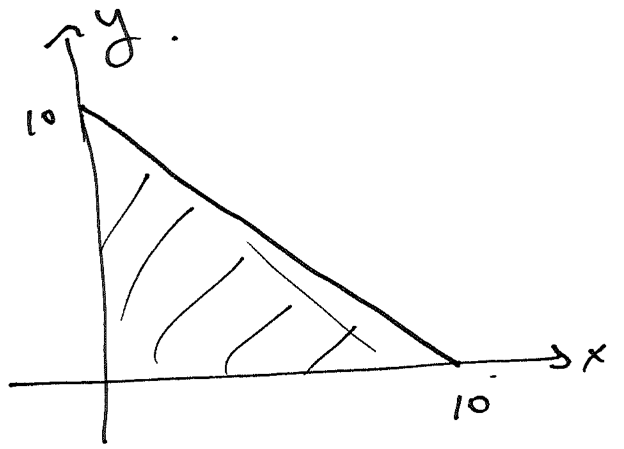
3) if $\det(\mathrm{Hess}_f(a)) < 0$ then a is a saddle point.

Global Extrema

Thm Let $f \in C^2(\Omega^\circ)$. Then every global extremum of f is either at a critical point in the interior of Ω or on the boundary of Ω.

Example. $f(x,y) = x^3 - 18x^2 + 81x + 12y^2 - 144y + 24xy$

on the set $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ (x+y) \leq 10\}$.



$$\nabla f = (3x^2 - 36x + 81 + 24y,\ 24y - 144 + 24x).$$

$$\nabla f = (0,0)$$

$$24y - 144 + 24x = 0 \Rightarrow 24y = 144 - 24x$$

$$3x^2 - 36x + 81 + 144 - 24x = 0.$$

$$3x^2 - 60x + 225 = 0 \Rightarrow \cancel{x = 15},\quad x = 5.$$

$$x^2 - 20x + 75 = 0$$

not in Ω.

$x = 5$, $y = 1$.

$f(5,1) = 548$.

We also need to check the boundary lines.

(a) $x=0$ (y-axis).

$f(x,y) = f(0,y) = g(y) = 12y^2 - 144y$.

This is a one variable extreme problem.

$g'(y) = 24y - 144 = 0 \Rightarrow y = 6$

$f(0,6) = -432$. At the end points $f(0,0) = 0$, $f(0,10) = -240$.

(b) $y=0$ $f(x,0) = x^3 - 18x^2 + 81x = h(x)$

$\Rightarrow h'(x) = 0 = 3x^2 - 36x + 81$

$x^2 - 12x + 27 = 0 = (x-9)(x-3)$

$\Rightarrow (3,0)$. $(9,0)$

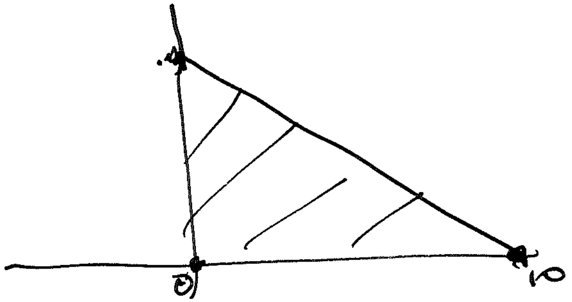
$f(3,0) = 108$

$f(9,0) = 0$. , At the end points $f(0,0) = 0$, $f(10,0) = 10$.

4-b

(3) on $y+x=10$.

$$f(x, 10-x) = x^3 - 18x^2 + 81x + 12(10-x)^2 + 144(10-x) + 24x(10-x).$$
$$= k(x)$$
$$k'(x) = 0 \Longrightarrow \cancel{x = 20}, \quad \cancel{x = -1}$$



$$f(5,1) = \underline{548}. \quad \text{max}$$
$$f(3,0) = 108$$
$$f(9,0) = 0$$
$$f(0,6) = \underline{-432} \quad \text{min}.$$

$$\boxed{f(10,0) = 10^3 - 18 \cdot 10^2 + 81 \cdot 10}$$

$$f(0,10) = 12 \cdot 10^2 - 144 \cdot 10.$$

$$\underline{f(10,0)} = \underbrace{-8 \cdot 100 + 81 \cdot 10}_{10} < 548.$$

Max at $(x, y) = (5,1)$

(4.2)

§7.6. Vector-valued functions

Let $\Omega \subset \mathbb{R}^n$

$$f : \Omega \longrightarrow \mathbb{R}^m$$
$$x \longmapsto \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

The differential theory of f is a straightforward extension of the theory of scalar fields. $f_i : \mathbb{R}^n \to \mathbb{R}$. are scalar fields $1 \leq i \leq m$.

Defn. : The function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is called differentiable in $x_0 \in \Omega$ if each component function $f_i : \Omega \to \mathbb{R}$ is differentiable.

__Defn.__ If f is differentiable, the matrix of partial derivatives

$$\nabla f = df := \left(\frac{\partial f_i}{\partial x^j} \right) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial x^2} & \cdots & \frac{\partial f_1}{\partial x^n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x^1} & \frac{\partial f_m}{\partial x^2} & & \frac{\partial f_m}{\partial x^n} \end{pmatrix}.$$

is called the __Jacobian__ matrix or __functional matrix__

Warning! Do not confuse this with the Hessian Matrix of a scalar field $f: \mathbb{R}^n \to \mathbb{R}$.

Remark. 1) f is differentiable in x_0 means

$$f(x) = f(x_0) + df(x_0)\cdot(x - x_0) + o(|x - x_0|)$$

For $f: \mathbb{R}^n \to \mathbb{R}^m$ differentiably could be defined similarly as follows.

f is differentiable in $x_0 \iff \exists$ linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim\limits_{x \to x_0} \dfrac{\| f(x) - f(x_0) - A(x - x_0) \|}{\| x - x_0 \|} = 0.$

2) If f is differentiable, the matrix of the linear map A is given by the Jacobian Matrix df.

__Ex.__ For every Matrix $A_{m\times n}$ every $v \in \mathbb{R}^m$.

The Affine linear map $f: \mathbb{R}^n \to \mathbb{R}^m$, $x \longmapsto Ax + v$

is differentiable with $df = A$.

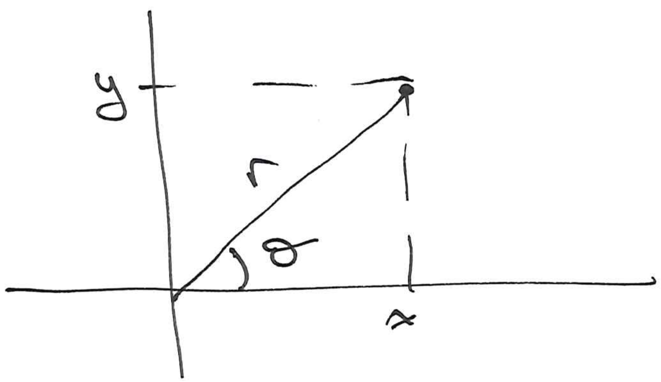
In the special case $f: \mathbb{R}^n \to \mathbb{R}^n$, $x \longmapsto x$

has $df = I_n$.

__Ex:__ $f: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \longmapsto \begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}$ $\qquad df = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$.

Ex. Polar coordinates.

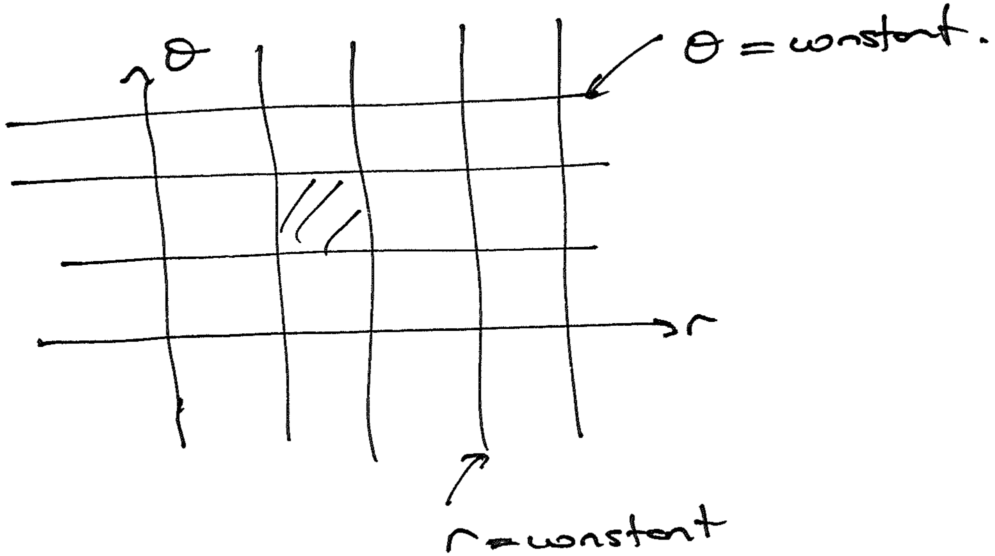
$$f : [0,\infty) \times [0, 2\pi) \longrightarrow \mathbb{R}^2$$
$$(r, \theta) \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

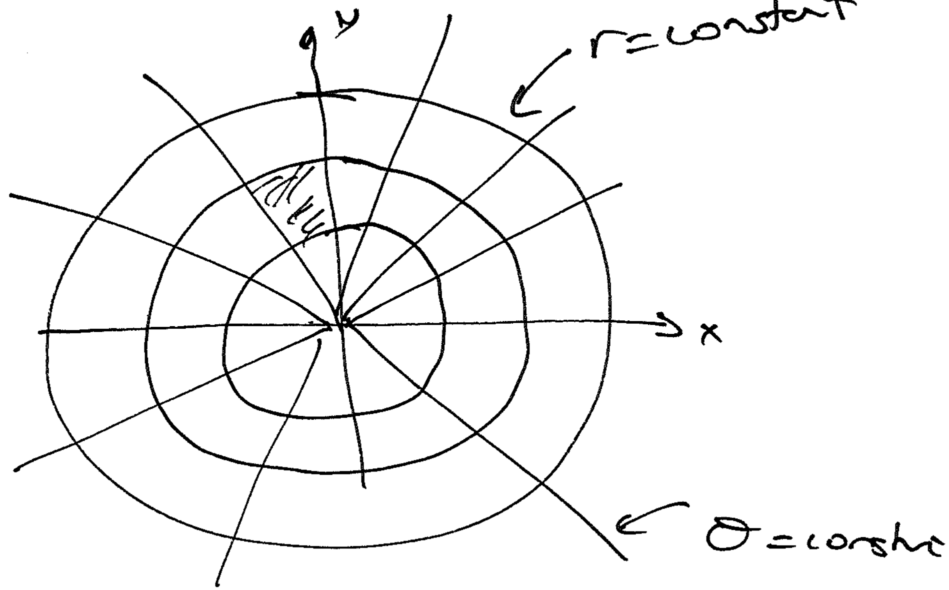


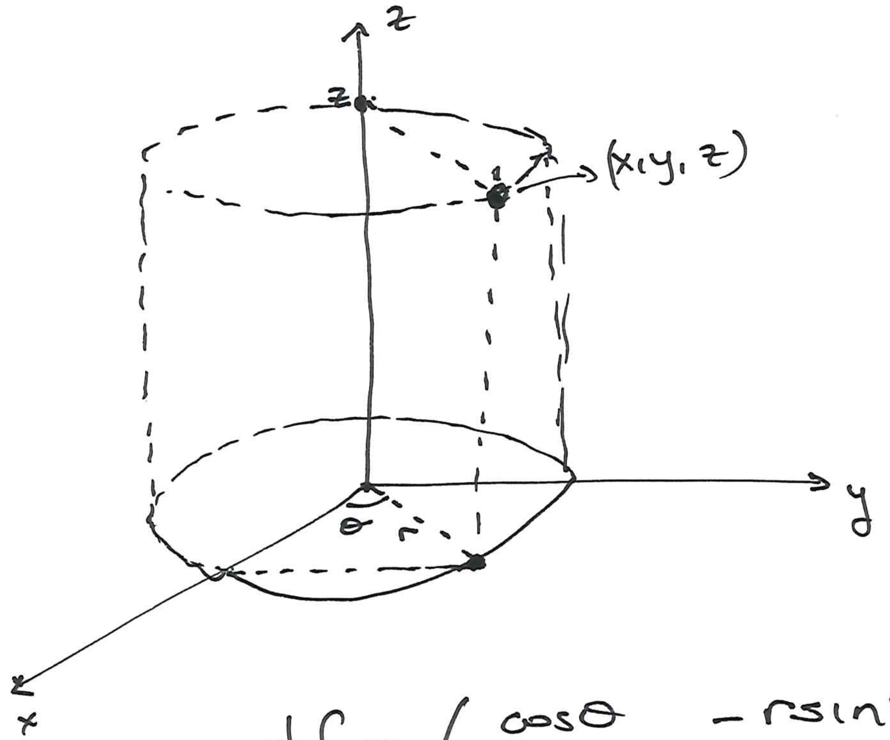
$$df(r,\theta) = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix}.$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det(df) = r(\cos^2\theta + \sin^2\theta) = r.$$


θ
θ = constant.
r
r = constant


y
r = constant
x
θ = constant



Ex : Cylindrical coordinates.

$$f = [0,\infty) \times [0,2\pi) \times \mathbb{R} \to \mathbb{R}^3$$

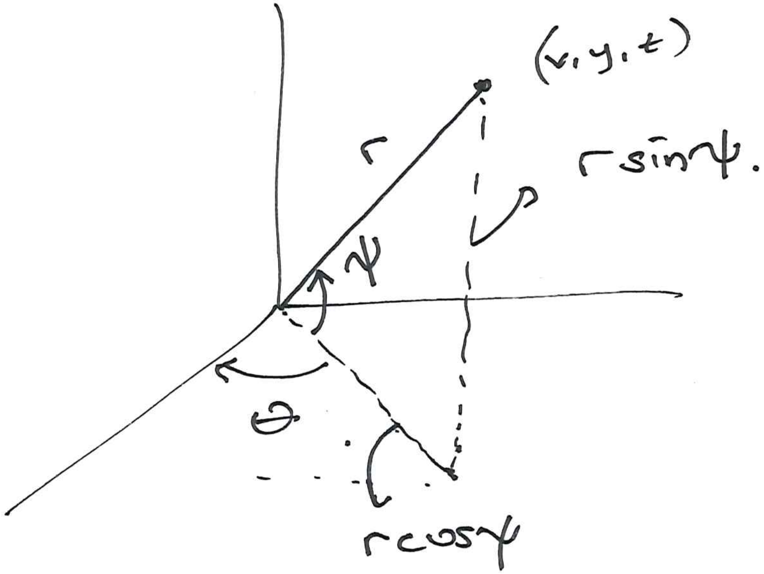
$$(r,\theta,z) \longmapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}.$$

$$df = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotational symmetry wrt z-axis

$$\det(df) = r$$

Ex: Spherical coordinates.



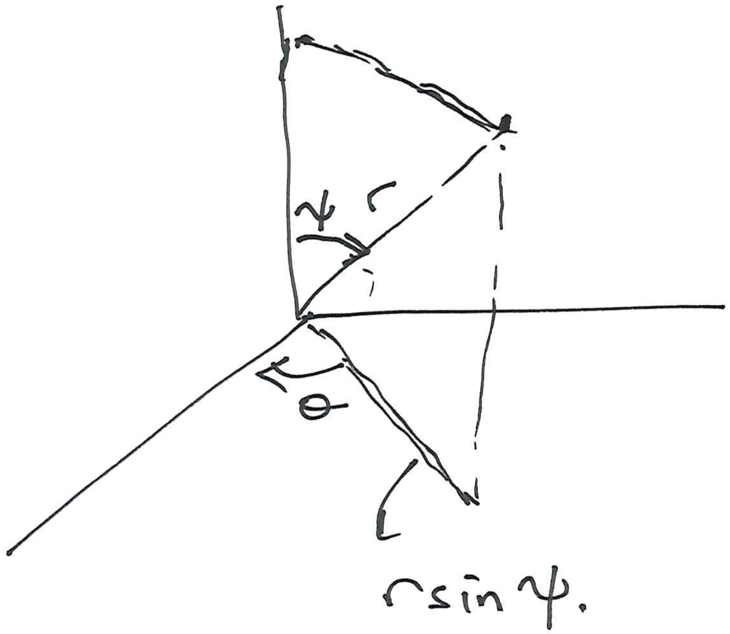
$$f = [0,\infty), [0,2\pi) \times \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \to \mathbb{R}^3.$$

$$r, \theta, \psi \longrightarrow \begin{pmatrix} r\cos\psi\cos\theta \\ r\cos\psi\sin\theta \\ r\sin\psi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$df = \begin{pmatrix} \cos\theta\cos\psi & -r\sin\theta\cos\psi & -r\cos\theta\sin\psi \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$\det(df) = r^2\cos\psi.$$

Remark. One could also use the following spherical coordinates.



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\psi\cos\theta \\ r\sin\psi\sin\theta \\ r\cos\psi \end{pmatrix}.$$

$$\det(df) = r^2\sin\psi.$$

For $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ we have the following differentiation rules.

Thm Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable in $x_0 \in \Omega$ and $\alpha \in \mathbb{R}$. Then the functions αf, $f+g$ as well as scalar product $f \cdot g = \sum_{i=1}^{n} f_i g_i$ in x_0 are differentiable and

1) $d(\alpha f)(x_0) = \alpha \, df(x_0)$

2) $d(f+g)(x_0) = (df)(x_0) + (dg)(x_0)$

3) $\underbrace{d(f \cdot g)(x_0)}_{1 \times n \text{ matrix}} = \underbrace{f(x_0)}_{1 \times m} \cdot \underbrace{dg(x_0)}_{m \times n} + \underbrace{g(x_0)}_{1 \times m} \cdot \underbrace{df(x_0)}_{m \times n}$.

$f \cdot g : \mathbb{R}^n \to \mathbb{R}$.

Thm (Chain rule) Let $g : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$ be diff in x_0, and $f : \mathbb{R}^\ell \longrightarrow \mathbb{R}^m$ diff. in $g(x_0)$. Then $f \circ g : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff. in x_0, and

$$\underbrace{d(f \circ g)(x_0)}_{m \times n} = \underbrace{\underbrace{(df)(g(x_0))}_{m \times \ell} \cdot \underbrace{dg(x_0)}_{\ell \times n}}_{m \times n}.$$

Ex: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

$$df = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x^2 + y^2 + z^2 \\ xyz \end{pmatrix}$$

$$dg = \begin{pmatrix} 2x & 2y & 2z \\ yz & xz & xy \end{pmatrix}$$

$$(f\circ g) : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} (x^2+y^2+z^2)^2 - x^2y^2z^2 \\ 2(x^2+y^2+z^2).\, xyz \end{pmatrix}.$$

$$d(f\circ g) = \begin{pmatrix} 2\,(x^2+y^2+z^2).2x - 2xy^2z^2 & * & * \\ \times & \times & \times \end{pmatrix}.$$

$$= df\,(g(x)) \cdot dg(x).$$

$$\begin{pmatrix} 2\,(x^2+y^2+z^2) & -2xyz \\ 2xyz & 2(x^2+y^2+z^2) \end{pmatrix} \begin{pmatrix} 2x & 2y & 2z \\ yz & xz & xy \end{pmatrix} = \begin{pmatrix} 4x(x^2+y^2+z^2) - 2xy^2z^2 & \cdot & \cdot \\ \times & \times & \times \end{pmatrix}$$

§ Line Integral.

We'll define the line integral (path integral) (wegintegral kurven integral) of a vector field along a curve.

Let γ be a vector valued function on an interval $[a, b] \subset \mathbb{R}$.

ie
$$\begin{aligned} \gamma : [a, b] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto (\gamma_1(t), \gamma_2(t) \ldots \gamma_n(t)) \end{aligned}$$

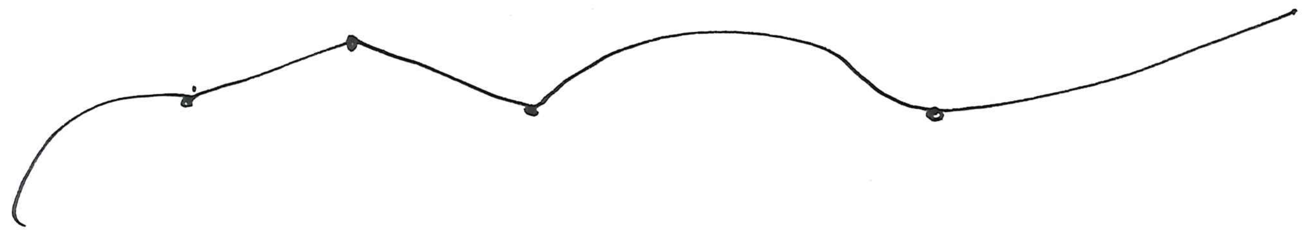
Then the values $\gamma(t)$ trace out a set of points in $\mathbb{R}^n$ which is called a curve (or the curve parametrized by γ.

Defn. Let $I = [a,b] \subset \mathbb{R}$ be a finite closed interval in $\mathbb{R}$. A function $\gamma : I \to \mathbb{R}^n$, $t \mapsto \gamma(t)$

which is continuous on I is called a continuous path in n-space. The path is called smooth if the derivative $\gamma'(t) = (\gamma_1'(t), \gamma_2'(t), \ldots, \gamma_n'(t))$ exists and continuous in the open interval (a,b).

$\gamma(t)$ is called a parametrization of the curve $\text{Image}(\gamma) \subset \mathbb{R}^n$.

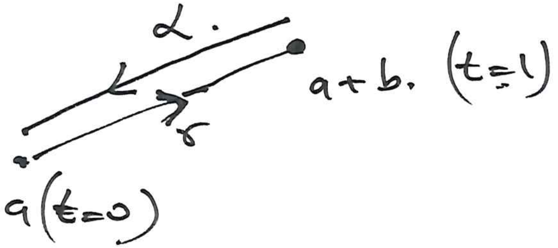
The path is called piecewise smooth if the interval $[a,b]$ can be partitioned into a finite number of subintervals $[a_0 = a, a_1], [a_1, a_2], \ldots, [a_{i-1}, a_i = b]$ in each of which the path is smooth.



Ex. 1) $\gamma : [0,1] \longrightarrow \mathbb{R}^3$

$t \longrightarrow (a_1 + b_1 t,\ a_2 + b_2 t,\ a_3 + b_3 t)$

Is the parametrization of the line segment in $\mathbb{R}^3$ through the point $a = (a_1, a_2, a_3)$ in the direction of $b = b_1, b_2, b_3$ in between the points a and $a+b$



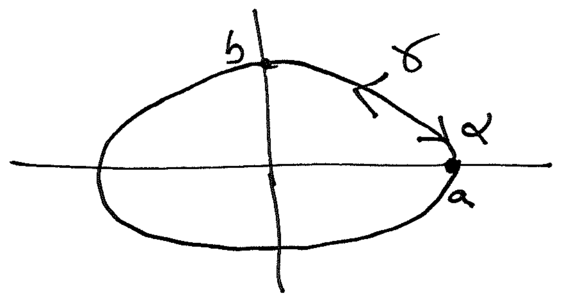
Note if $t \in \mathbb{R}$ then we get the whole line
$r(t) = \vec{a} + \vec{b}t$

Note the curve

$$\alpha : [0,1] \longrightarrow \mathbb{R}^3$$
$$t \longrightarrow (a_1 + b_1(1-t),\ a_2 + b_2(1-t),\ a_3 + b_3(1-t))$$

traces the same line segment but in the opposite direction

(2) $\gamma : [0, 2\pi] \longmapsto (a\cos t,\ b\sin t)$



is a parametrization of the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

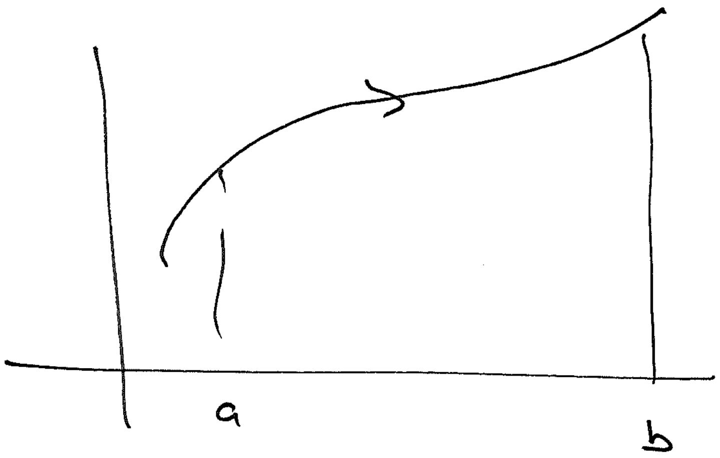
If $a = b$, then $\gamma(t)$ gives a parametrization of circle of radius a centered at origin.

$\alpha : [0, 2\pi] \longrightarrow (a\cos(2\pi - t),\ b\sin(2\pi - t)) = (a\cos t,\ -b\sin t)$ (20)

traces the same ellipse in the opposite direction

(3) $f: \mathbb{R} \to \mathbb{R}$ be a continuous function

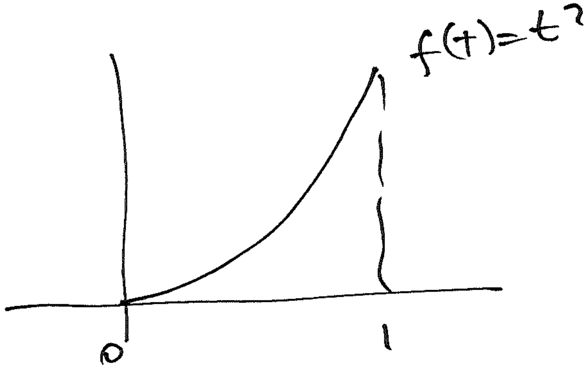
Then its graph is a curve in $\mathbb{R}^2$



and $\gamma: [a, b] \to \mathbb{R}^2$

$t \mapsto (t, f(t))$

is a parametrization of this curve.

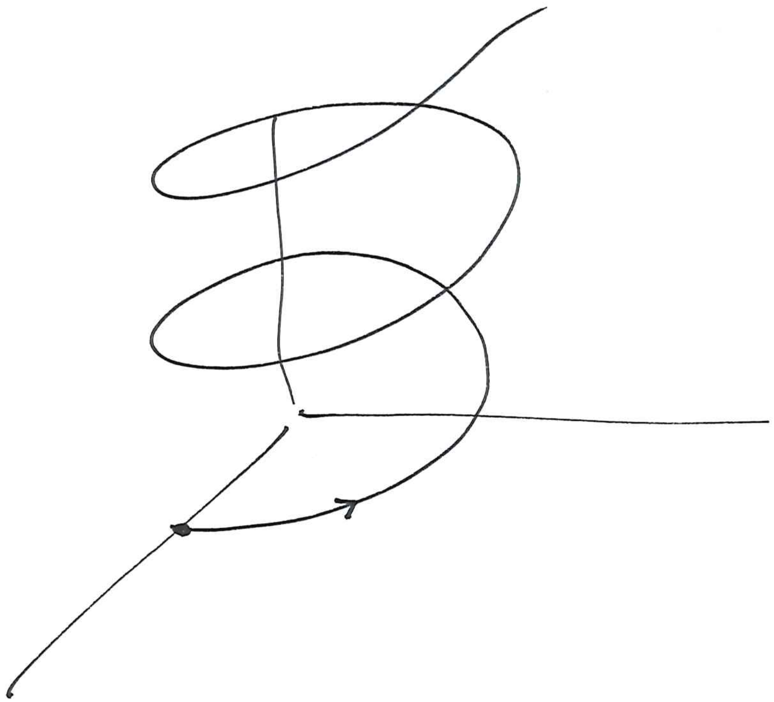


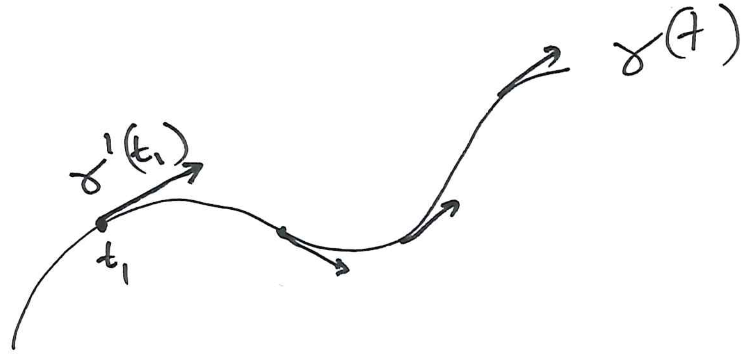
$\gamma: [0,1] \to \mathbb{R}^2$

$t \to (t, t^2)$

<u>Ex:</u> $\gamma : \cancel{\mathbb{R}}\ [0,\infty) \longrightarrow \mathbb{R}^3$

$t \longrightarrow (a\cos t,\ a\sin t,\ t)$





Note $\gamma'(t)$ describes the tangent vector to the curve at each point

Defn Let $v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a vector field

$\gamma \subset \Omega$ a curve with parametrization

$$\gamma : I \longrightarrow \Omega \subset \mathbb{R}^n \quad , \quad I = [a, b]$$
$$t \longrightarrow \gamma(t) .$$

Then the line integral of v along γ

is defined as $\displaystyle\int_a^b v(\gamma(t)) \cdot \gamma'(t)\, dt$

and is denoted by $\displaystyle\int v\, d\gamma \ , \ \int_\gamma v \cdot ds \ , \ \int v(\gamma)\, d\gamma .$

$ds = \gamma'(t)\, dt$ is called the line element.

Other notations for line integrals.

When v and γ are expressed in terms of their components

$$v = v_1, v_2 \ldots v_n \qquad v : \mathbb{R}^n \to \mathbb{R}^n$$

$$\gamma : (\gamma_1, \gamma_2, \ldots \gamma_n) \qquad \gamma : [a, b] \to \mathbb{R}^n$$

$$\int_\gamma v \cdot ds = \int v \cdot d\gamma$$ is also written as

$$\int_a^b v_1 \, d\gamma_1 + v_2 \, d\gamma_2 \ldots + v_n \, d\gamma_n .$$

In the special case of 2 dimensions $v = v_1, v_2)$

$$v = \mathbb{R}^2 \to \mathbb{R}^2 \qquad \gamma : I \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad t \to \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$x = \gamma_1(t) \qquad y = \gamma_2(t)$

$dx = \gamma_1'(t)\,dt \qquad dy = \gamma_2'(t)\,dt$

$$\int_\gamma v\,ds = \int_a^b \begin{pmatrix} v_1(\gamma_1(t), \gamma_2(t)) \\ v_2(\gamma_1, \gamma_2) \end{pmatrix} \cdot \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} dt$$

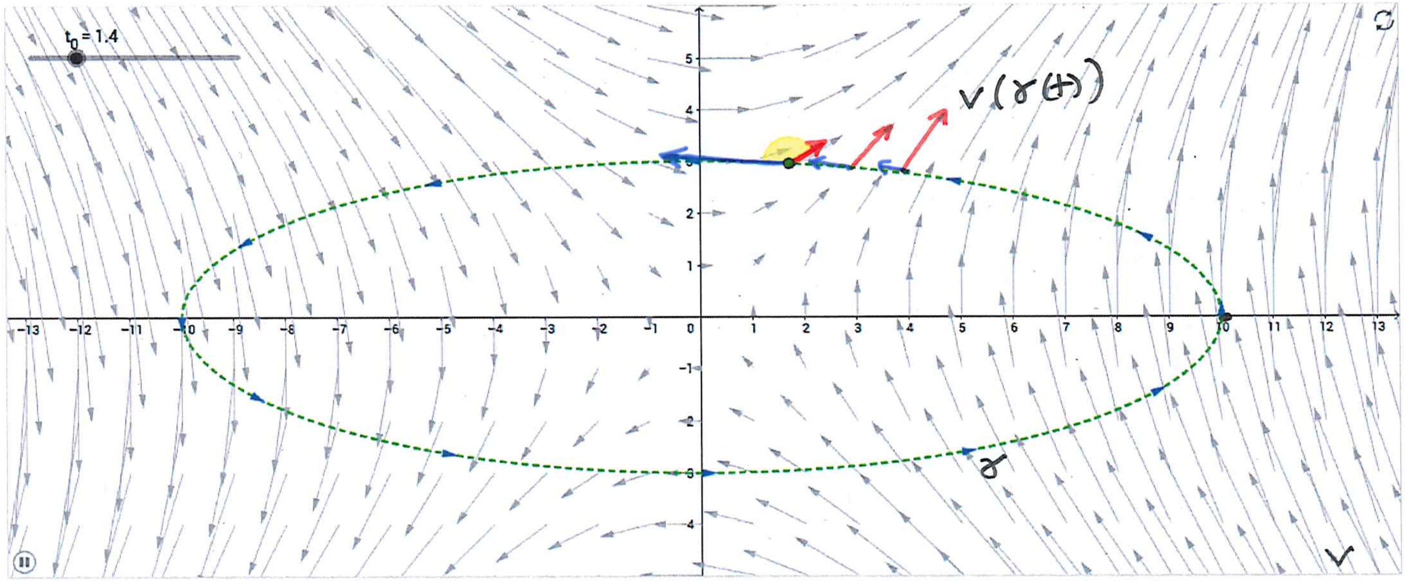
$$= \int v_1(\gamma(t)) \cdot \gamma_1'(t)\,dt + v_2(\gamma(t)) \cdot \gamma_2'(t)\,dt$$

$$= \int_a^b v_1(x, y)\,dx + v_2(x, y)\,dy$$

Line Integral of a Vector Field in 2-Space

This worksheet illustrates the integral of a vector field along a closed curve in the plane.

$V: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



$$\int_a^b \underbrace{v(\gamma(t))}_{\text{red vector}} \cdot \overbrace{\gamma'(t)}^{\text{blue vector}} \, dt$$