Let D be a thin plate in $\mathbb{R}^2$ with matter distributed with density $f(x,y)$ (mass/unit area)

The total mass of D is $m(D) = \iint_D f(x,y)\,dx\,dy$

The average density is $\frac{m(D)}{\text{Area}(D)} = \left(\iint_D f(x,y)\,dx\,dy\right) / \iint_D dx\,dy$

Center of mass of D is $(\bar{x}, \bar{y})$ where

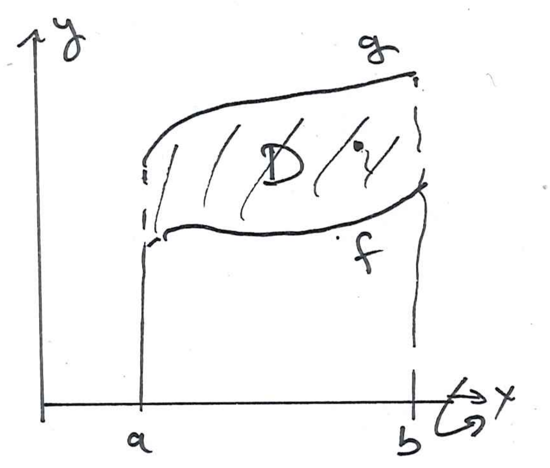
$$\bar{x} = \bar{x}(D) = \frac{1}{m(D)} \iint_D x\, f(x,y)\,dx\,dy$$

$$\bar{y} = \bar{y}(D) = \frac{1}{m(D)} \iint_D y\, f(x,y)\,dx\,dy$$

If density $f(x,y) \equiv 1$, the center of mass is called the centroid and $(\bar{x}, \bar{y}) = \dfrac{\left(\iint x\,dx\,dy,\ \iint y\,dx\,dy\right)}{\iint_D dx\,dy}$

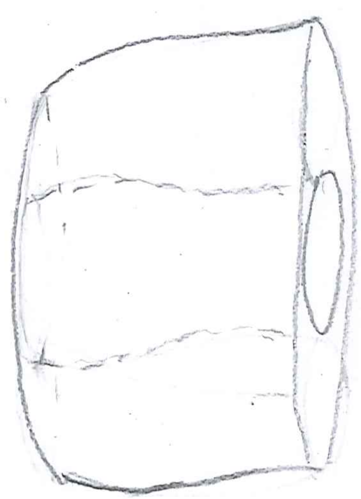
Ex: Centroid of a circular region is located at the center

Thm of Pappus



Let D be the region lying between the graphs of 2 continuous functions f and g over an interval $[a,b]$, where $0 \le f \le g$.
Let S be the solid of revolution generated by rotating D about the x-axis, let V be the volume of S, and $(\bar{x},\bar{y})$ the centroid of D. Then

$$V(S) = 2\pi \bar{y} A(D)$$



Pf. $V = \pi \int_a^b (g(x))^2 - (f(x))^2 \, dx$

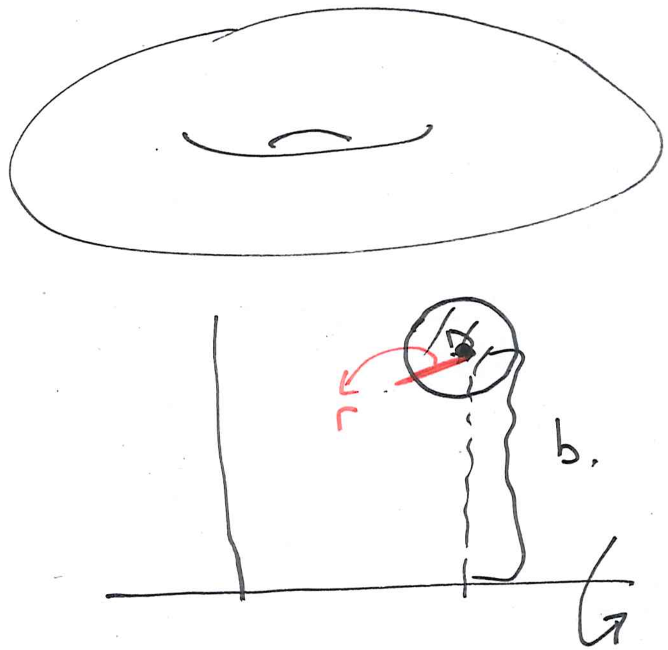
$$\bar{y} = \frac{1}{A(D)} \iint_D y \, dx \, dy.$$

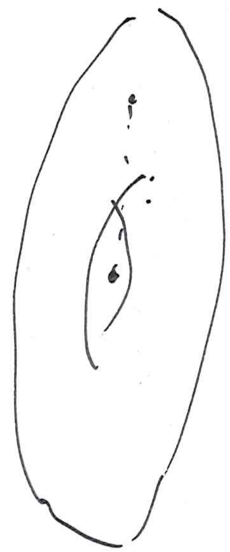
$$= \frac{1}{A(D)} \int_a^b \int_{f(x)}^{g(x)} y \, dy \, dx$$

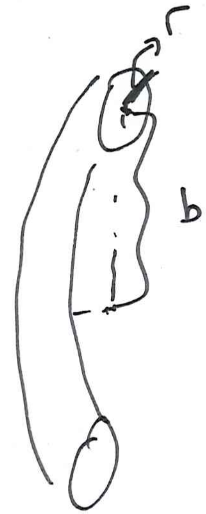
$$= \frac{1}{A(D)} \int_a^b \left(\frac{y^2}{2} \Big|_{f(x)}^{g(x)} \right) dx = \frac{1}{A(D)} \frac{1}{2} \int_a^b g^2(x) - f^2(x) \, dx.$$

Hence $V = 2\pi A(D) \, \bar{y}(D)$

Ex. Volume of a Torus (Donut).





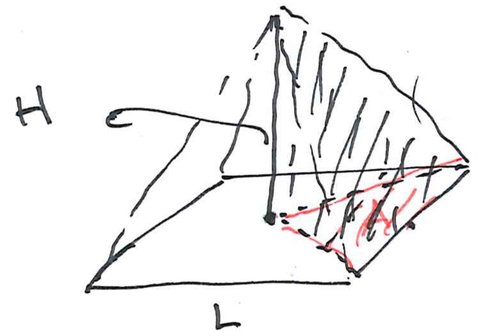


To find the volume of a torus that is obtained by rotating about x-axis a circular region D of radius r, lying above the x-axis with a distance from its center to x-axis being. b, By Pappus' thm $\quad \text{vol}(\text{Torus}) = 2\pi\, \bar{y}(D) . A(D)$

For a circular region the centroid is at the center. Hence $y(\bar{D})$ is the distance of the center of D to the x-axis, i.e. b.

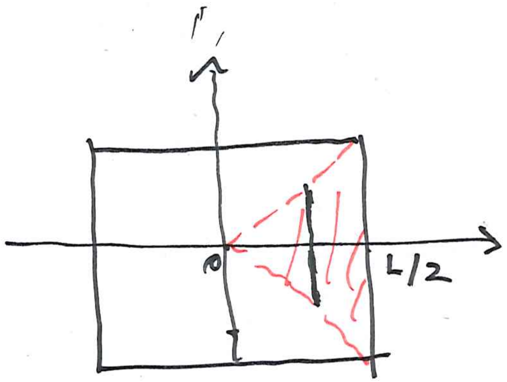
Hence $\text{vol}(\text{Torus}) = 2\pi b (\pi r^2) = 2\pi^2 r^2 b$.

Ex: Find the volume of a pyramide with square base of length L, and height H.



From school $V = \frac{1}{3} L^2 H$.

Due to symmetry we can find $\frac{1}{4}$ of the volume and multiply by 4.

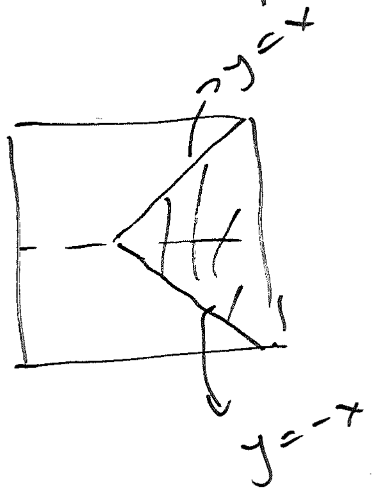


We want to write the right side, which is a plane, as the graph of a function. Note the height from this plane to the base gives this function which does depend on x (and only on x)

The height is linear and it is 0 if $x = \frac{L}{2}$ and it is H if $x = 0$.

Hence $z = h(x,y) = H - \frac{2H}{L}x = H\left(1 - \frac{2}{L}x\right)$.

Hence $\frac{1}{4}V = \iint_A h(x,y)\,dx\,dy = \iint \left(H - \frac{2H}{L}x\right) dy\,dx$



$$= \int_0^{L/2}\int_{y=-x}^{y=x} \left(H - \frac{2H}{L}x\right) dy\,dx$$

$$= \cdots\cdots = \frac{L^2H}{12}.$$

Hence $V = \frac{1}{3}L^2H$.

Green's theorem in the Plane.

Recall the following 2 fundamental theorems about integration:

① Fund. thm of 1. dim'l integral calculus.

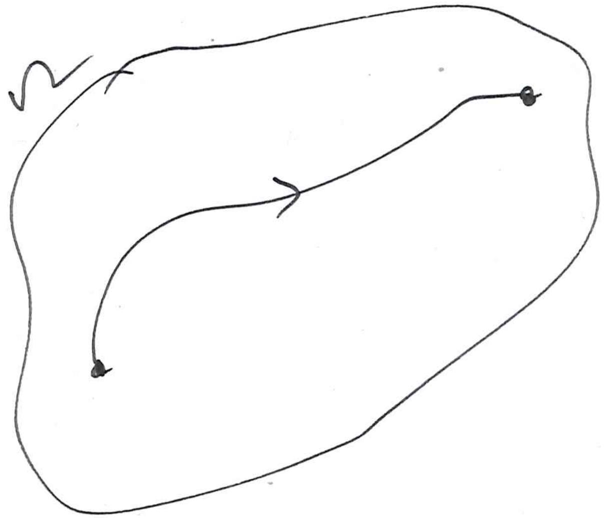
If F is a primitive of f (ie $F' = f$, $dF = f(x)dx$).

then $$\int_a^b f\,dx = \int_a^b dF = F(b) - F(a).$$

$$\int_{[a,b]} f\,dx = F(b) - F(a).$$

(2) Fund. thm for line integrals

Let γ be a smooth curve $\gamma : [a, b] \to \mathbb{R}^2$ in $\mathbb{R}^2$ with end points $\gamma(a)$, $\gamma(b)$



Then for any C^1 scalar field $F : \mathbb{R}^2 \to \mathbb{R}$ defined on a connected set $\Omega \subset \mathbb{R}^2$ containing γ, we have

$$\int_\gamma \nabla F \, d\gamma = F(\gamma(b)) - F(\gamma(a)).$$

ie. if V is a conservative vector field, ie $V = \nabla F$ for some F then $\int_\gamma V \, dr = F(\gamma(b)) - F(\gamma(a))$.

The integral of V along γ (if V is conservative) depends only on the difference of the values of F at the end pts.

This raises the following question.

Question

Do similar formulas hold over surfaces and higher dimensional objects?

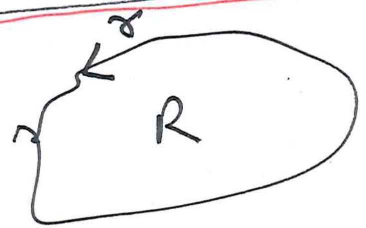
i.e. If the integral of the analog of a gradient (∇F, $dF = f$) (sometimes called an exact differential) over a geometric shape $\mathcal{M}$ depends only on the integral of its "primitive" on the boundary of $\mathcal{M}$.

Our goal is to establish the simplest instance of such a phenomenon for regions in $\mathbb{R}^2$.

Green's thm. It expresses a double integral over a region R as a line integral taken along a closed curve forming the boundary of the region R.

The most common form of Green's identity.

$$\boxed{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \oint_\gamma P\, dx + Q\, dy}$$



In the above identity two types of assumptions are implicit.

i) There are assumptions on the functions P, Q the components of the vector field

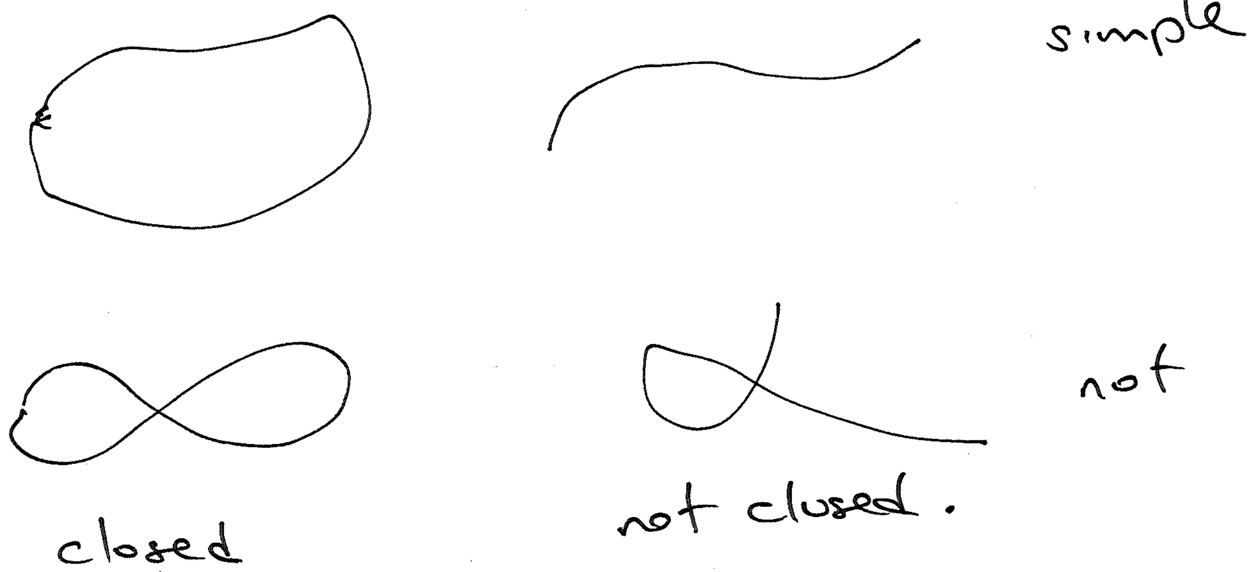
Namely $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ exist in the region R

The usual assumption is that P, Q are continuously differentiable. This implies $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ exist and continuous and hence $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ is continuous and hence integrable over R.

2) There are conditions implicitly imposed on the region R and its boundary curve γ.

Namely the curve is closed and "simple"

Simple means that it does not intersect itself.



A simple closed curve is called a <u>Jordan curve</u>.

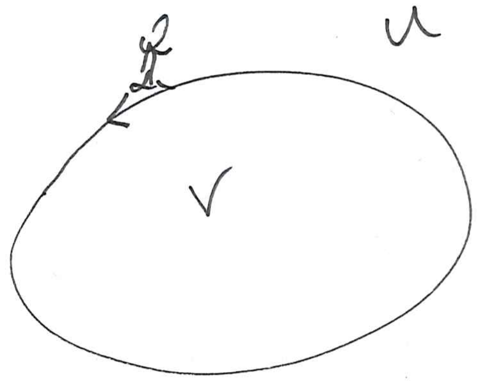
<u>Thm.</u> (Jordan curve theorem). Let γ be a Jordan curve in $\mathbb{R}^2$. Then $\exists$ connected open sets U, V in $\mathbb{R}^2$ such that

1) U, V, γ are pairwise mutually disjoint.

2) $\mathbb{R}^2 = U \cup V \cup \gamma$

Any Jordan curve γ separates the plane into 2 distinct connected region with γ as a common boundary. 2 regions U, V are called the interior (inside) and exterior (outside) of γ.

<u>Lemma.</u> In the above situation exactly one of U and V is bounded. This bdd one is called the interior.

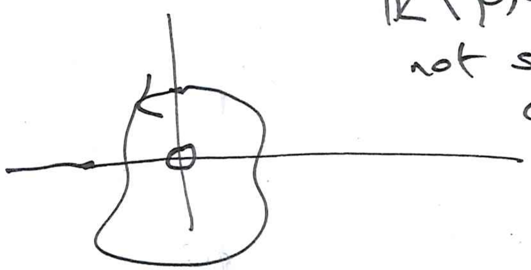


A Jordan curve is oriented "positively" if the interior lies to the left as we traverse the curve.

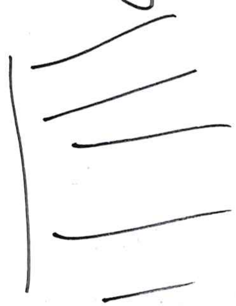
<u>**Defn**</u> ~~Finally~~ An open connected region $\Omega \subset \mathbb{R}^2$ is called simply connected if and only if every Jordan curve γ in Ω can be continuously deformed to a point without crossing itself in the process.

~~or~~ equivalently for any Jordan curve $\gamma \in \Omega$ the interior of γ lies completely in Ω.

$\mathbb{R}^2 \setminus \{0,0\}$ not simply connected



$\mathbb{R}^+$ is simply connected.



Thm (Green). Let $f(x,y) = (P(x,y), Q(x,y))$ be a vector field that is continuously differentiable on an open simply connected set $\Omega \subset \mathbb{R}^2$. Let γ be a piecewise smooth closed curve and let R be the union of γ and its interior

Then we have

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint_\gamma P\, dx + Q\, dy .$$

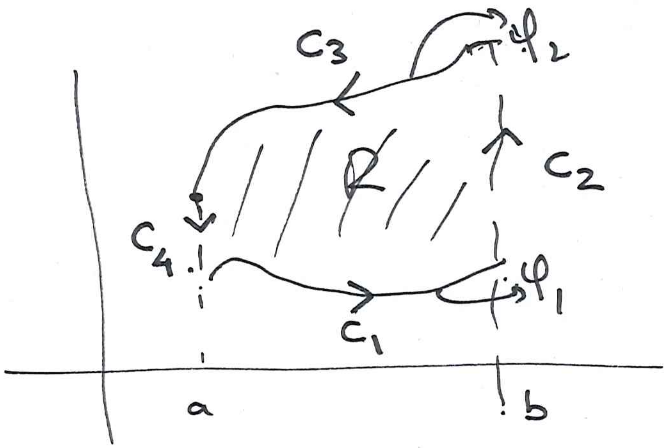
where the line integral is taken around γ in the ccw direction

---

Note Since Ω is simply connected, $R \subset \Omega$, Pf: We'll give the proof for special regions.

Namely assume R is of type I

$$R = \{ a \le x \le b, \quad \varphi_1(x) \le y \le \varphi_2(x) \mid \varphi_1, \varphi_2 \text{ are continuous on } [a,b] \}.$$



C_1 : the graph of φ_1

C_2 : $x = b$ $\quad \varphi_1(b) < y < \varphi_2(b)$

C_3 graph of φ_2

C_4 : $x = a$ $\quad \varphi_1(a) \le y < \varphi_2(a)$.

__Rk__. The Green's identity is equivalent to the two formulas

$$\iint_R \frac{\partial Q}{\partial x}\, dx\, dy = \oint_\gamma Q\, dy \qquad (*)_1$$

and

$$-\iint_R \frac{\partial P}{\partial y}\, dx\, dy = \oint_\gamma P\, dx \qquad (*)_2 .$$

Indeed if $(*)_1$ and $(*)_2$ holds, so does

$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy = \oint P\, dx + Q\, dy . \qquad (*)$$

and if $(*)$ holds for any $f = (P, Q)$, taking $f = (P, 0)$ or $(0, Q)$ gives $(*)_1$, $(*)_2$.

We'll prove $-\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \oint_C P\,dx$.

Clearly on C_2, C_4 $\int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$.
(x does not change).

Hence $-\oint_C P\,dx = -\left[\int_{C_1} P\,dx + \int_{C_3} P\,dx\right.$

$$= -\left[\int_a^b P((x, \varphi_1(x))\,dx + \int_b^a P(x, \varphi_2(x))\,dx.\right.$$

$$= \int_a^b \left[P(x, \varphi_2(x)) - P(x, \varphi_1(x))\right] dx$$

By assumption P is C^1, $\frac{\partial P}{\partial y}$ is continous.

$$\iint_R \frac{\partial P}{\partial y}\, dx\, dy = \int_a^b \underbrace{\left(\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\, dy \right)}_{\text{Fund. thm of } 1^{\text{'}}\text{dnl calculus.}} dx .$$

$$\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}\, dy = P(x, \varphi_2(x)) - P(x, \varphi_1(x)).$$

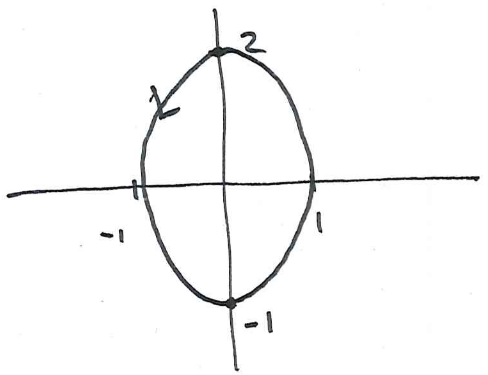
Hence $\displaystyle \iint_R \frac{\partial P}{\partial y}\, dx\, dy = \int_a^b P(x, \varphi_2(x)) - P(x, \varphi_1(x))\, dx$

Hence $\displaystyle -\iint_R \frac{\partial P}{\partial y}\, dx\, dy = \oint_\gamma P\, dx$

Q- part is similar.

Ex: let v the vector field $v(x,y) = (y+3x, y-2x)$ and γ be the curve defined by the ellipse $4x^2 + y^2 = 4$.

$$\int_\gamma v\,dr = \int_\gamma (y+3x)\,dx + (y-2x)\,dy.$$



a parametrization of $\gamma : [0, 2\pi] \to \mathbb{R}^2$

$t \mapsto (\cos t, 2\sin t)$

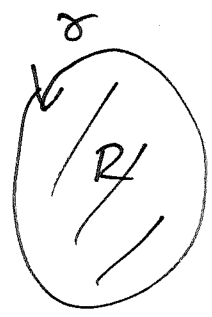
$\gamma'(t) = (-\sin t, 2\cos t)$

$$\int_\gamma v\,ds = \int_0^{2\pi} (2\sin t + 3\cos t)(-\sin t) + (2\sin t - 2\cos t) \cdot 2\cos t\,dt$$

$$= \int_0^{2\pi} -2 + \frac{\sin 2t}{2} \ldots \quad dt = \ldots = -6\pi$$

$+ 2\cos^2 t$

On the other hand using Green's thm



$$\int_{\gamma} v\, ds = \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{rot(v)} dx\,dy.$$

$$\gamma \downarrow$$

$$(\underbrace{y+3x}_{P}, \underbrace{y-2x}_{Q})$$

$$= \iint_{R.} (-2-1)\, dx\,dy = -3 \iint_R dx\,dy$$

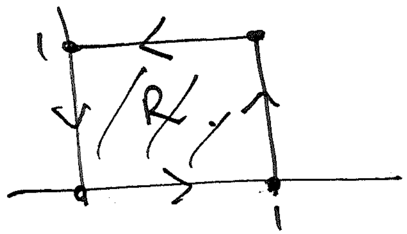
$$= -3 \underbrace{(\text{area of the ellipse})}_{\pi\, 2.1.}$$

$$= -6\pi.$$

$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$ has area πab.

Ex ②. $\int_{\gamma} (5-xy-y^2)\,dx - (2xy-x^2)\,dy$ where R is the square with corners at $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ traced in ccw direction

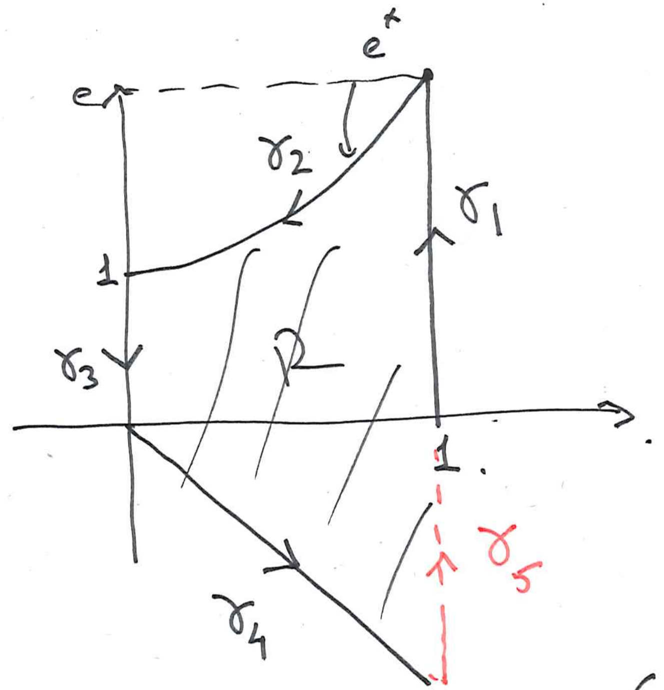
$\gamma = \partial R$.



Using Greens thm $\int_0^1\int_0^1 \Big(\underbrace{\frac{d}{dx}(-2xy+x^2) - \frac{\partial}{\partial y}(5-xy-y^2)}_{\mathrm{rot}(v)}\Big)\,dx\,dy$.

$$= \int_0^1\int_0^1 \left[(2x-2y) - (-2y-x)\right] dx\,dy = \int_0^1\int_0^1 3x\,dx\,dy = \dots 3/2.$$

Ex. (2010 exam question): Calculate the line integral $\int_\gamma v\,ds$ where γ is the path given in the picture, and $v(x,y) = (xy^2, -y)$



$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$.

1. We'll first calculate by calculating the line integrals over each curve $\gamma_1, \ldots, \gamma_4$.

2. Use Green's thm to write

$$\int_{\gamma_5} + \int_{\gamma} = \iint_R$$

$\gamma_1(t) : (1, t) \qquad 0 \leq t \leq e.$

$V = (xy^2, -y)$

$\gamma_2(t) : (t, e^t) \qquad t$ goes from 1 to 0.

$\gamma_3(t) = (0, t) \qquad t$ " " 1 to 0.

$\gamma_4(t) = (t, -t) \qquad t$ from 0 to 1.

$$\int_\gamma V\, ds = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}.$$

$$= \int_0^e (t^2, -t) \cdot (0,1)\, dt + \int_1^0 (te^{2t}, -e^t) \cdot (1, e^t)\, dt$$

$$+ \int_1^0 (0, -t) \cdot (0,1)\, dt + \int_0^1 (t^3, t) \cdot (1, -1)\, dt.$$

$$= \int_0^e -t\, dt + \int_1^0 te^{2t} - e^{2t}\, dt + \int_1^0 -t\, dt + \int_0^1 t^3 - t\, dt.$$

$$= - - - - - - - - = -\frac{e^2}{4} - \frac{1}{2}$$

2. or We can also use Green's thm by adding $\gamma_5(t) = (1, t)$ + goes from -1 to 0.

$$\int_{\gamma+\gamma_5} v \, ds = \iint_R (\text{rot } v) \, dx \, dy.$$

$v = (xy^2, -y)$

$$= \int_0^1 \int_{-x}^{e^x} \left(\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(xy^2) \right) dy \, dx.$$

$= \ldots =$