Let $v : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a vector field

$\gamma \in \Omega$ a curve with parametrization

$$\gamma : I \longrightarrow \Omega \quad , \quad I = [a, b] \text{ an interval}$$
$$t \longmapsto \gamma(t)$$

The line integral of v along γ is defined as

$$\int_a^b v(\gamma(t)) \cdot \gamma'(t)\, dt$$

denoted by $\int_\gamma v\, ds$ or $\int_\gamma v\, d\gamma$

In dim 2 :

$$\int_\gamma V\, ds = \int_\gamma V_1\, dx + V_2\, dy$$

$$V : \mathbb{R}^2 \to \mathbb{R} \qquad \gamma : I \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1(x,y) \\ V_2(x,y) \end{pmatrix} \qquad t \to (x(t), y(t))$$

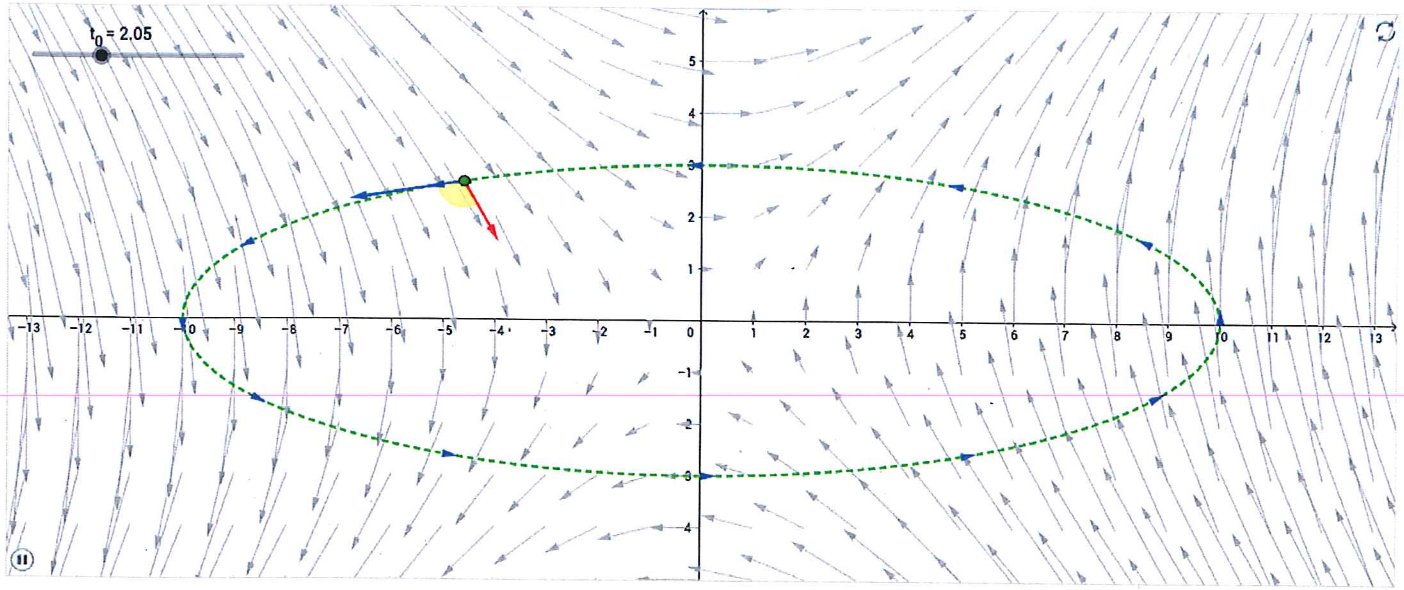
An expression of the form

$$\lambda(x^1 \ldots x^n) := V_1(x^1 \ldots, x^n)\, dx^1 + V_2(x^1 \ldots x^n)\, dx^2 + \ldots + V_n(x^1 \ldots x^n)\, dx^n$$

Is called a <u>differential 1-form</u> on $\Omega \subset \mathbb{R}^n$.

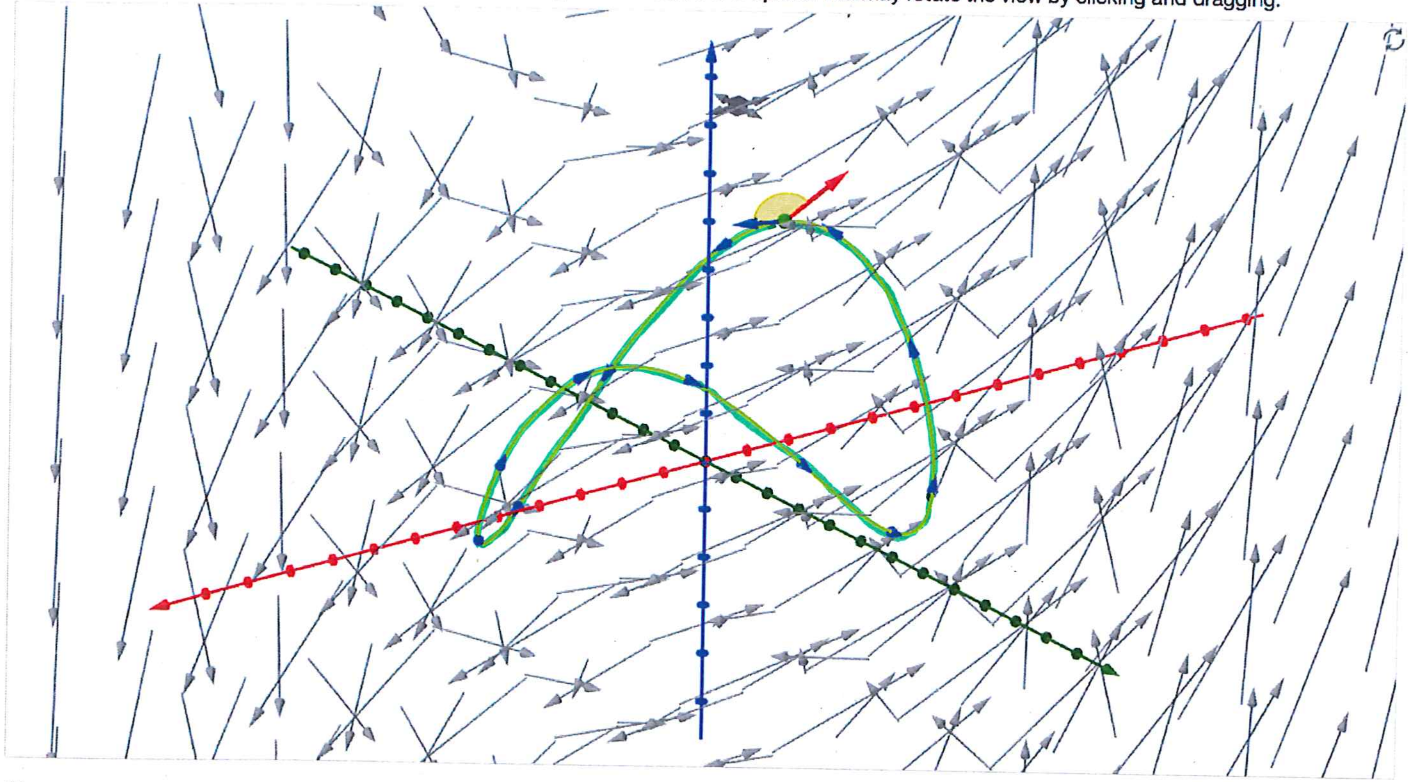
Line Integral of a Vector Field in 2-Space

This worksheet illustrates the integral of a vector field along a closed curve in the plane.



Line Integral of a Vector Field in 3-Space

This worksheet illustrates the integral of a vector field along a closed curve in 3-space. You may rotate the view by clicking and dragging.



What does the (animated) blue arrow represent?

Type your answer here...

The instantaneous direction vector along the curve.

What does the (animated) red arrow represent?

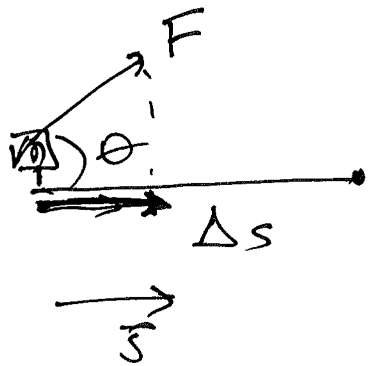
Type your answer here...

Why do we define such an integral?

An example ~~of~~ from physics.

Assume a point mass m moves under the influence of a force field $F: \mathbb{R}^2 \to \mathbb{R}^2$

When the point mass moves due to a constant force along a line. If it is moved a distance

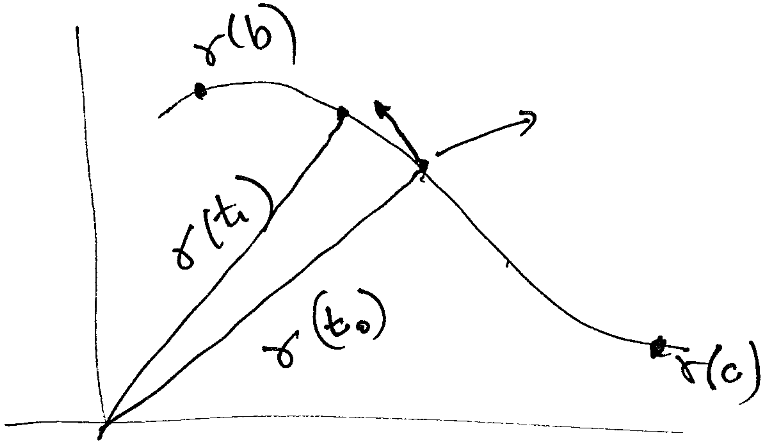


Δs, the amount of work done is given by

$$W = \vec{F} \cdot \vec{s} = \|F\| \|s\| \cos\theta.$$
$$= \|s\| \underbrace{\|F\| \cos\theta}_{\text{amount of Force in the direction of } s}.$$

Now assume it is moved along a curve and under the influence of a Forcefield which changes from point to point.

$\gamma : [a, b] \to \mathbb{R}^2$



Assume $F(x, y) = (P(x,y), Q(x,y))$.

$\gamma(t) = (x(t), y(t))$

$\Delta w = \vec{F} \cdot \vec{\Delta \gamma}$.

$= \left(\text{Component of } F \text{ along the curve}\right)\left(\text{travelled distance along the curve}\right)$.

To calculate total work, we divide the curve into small pieces (subcurves) partition the interval $[a, b]$ into $t_0 = a, t_1, \dots t_n = b$.

$$\Delta \gamma_i = \gamma(t_{i+1}) - \gamma(t_i) = \frac{\Delta \gamma}{\Delta t} \cdot \Delta t .$$

$$\Delta w_i = F(x(t_i), y(t_i)) \cdot \Delta \gamma_i$$

Then add together these small Δw_i to find the total work.

$$W \approx \sum_{i=1}^{n} \Delta w_i = \sum_{i=1}^{n} F(\gamma(t_i)) \cdot \frac{\Delta \gamma_i}{\Delta t} \Delta t$$

As the length of the portion ~~[illegible]~~ Δt small

i.e. $n \to \infty$

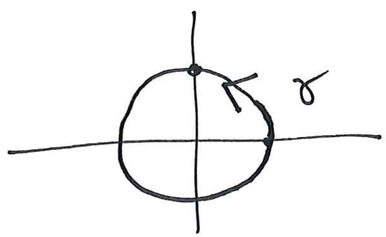
$$W = \int_a^b F(\gamma(t))\, \gamma'(t)\, dt .$$

Examples. (1) $F(x,y) = (-y, x)$.

$\gamma(t) = (\cos t, \sin t)$ $\quad t \in (0, 2\pi)$ $\quad \gamma'(t) = (-\sin t, \cos t)$

$$\int_\gamma F\,ds = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\,dt$$

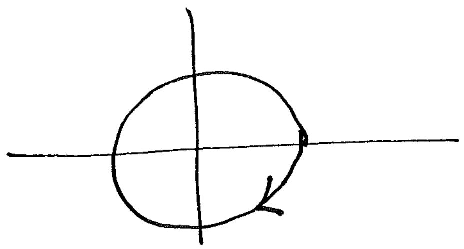
$$= \int_0^{2\pi} F(\cos t, \sin t) \cdot (-\sin t, \cos t)\,dt$$



$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} \underbrace{(\sin^2 t + \cos^2 t)}_{1}\,dt = 2\pi.$$

If we take the curve $\tilde{\gamma}(t) = (\cos t, -\sin t)$



$$\int_{\tilde{\gamma}} F \cdot ds = \int_0^{2\pi} F(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t)\, dt$$

$$= \int_0^{2\pi} (\sin t, \cos t) \cdot (-\sin t, -\cos t)\, dt$$

$$= -\int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= -2\pi.$$

Ie. change of orientation in the curve, changes the sign of the line integral.

Ex 2.

$$v : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x, y, z) \longmapsto (-y, x, z^2)$$

$\gamma(t) = (\cos t, \sin t, t) \qquad 0 \le t \le 2\pi \qquad \gamma'(t) = (-\sin t, \cos t, 1)$

$$\int_\gamma v \, ds = \int_0^{2\pi} v(\gamma(t)) \cdot \gamma'(t) \, dt$$

$$= \int_0^{2\pi} v((\cos t, \sin t, t)) \cdot (-\sin t, \cos t, 1) \, dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t, t^2) \cdot (-\sin t, \cos t, 1) \, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t + t^2) \, dt = \int_0^{2\pi} 1 + t^2 \, dt = t + \frac{t^3}{3} \Big|_0^{2\pi}$$

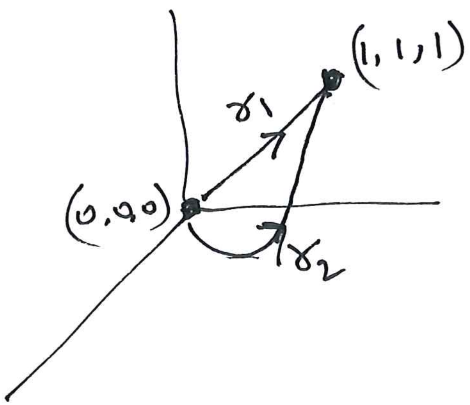
$$= 2\pi + \frac{(2\pi)^3}{3}.$$

Ex 3.

$$F = \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto (y^2, xz, 1).$$

$$\gamma_1(t) = t \longmapsto \begin{cases} x(t) = t \\ y(t) = t \\ z(t) = t \end{cases} \qquad 0 \le t \le 1$$

$$\gamma_2(t) = t \longmapsto \begin{cases} x(t) = t \\ y(t) = t^2 \\ z(t) = t^3 \end{cases} \qquad 0 \le t \le 1$$



$$\int_{\gamma_1} F\, ds = \int_0^1 F(\gamma_1(t))\, \gamma_1'(t)\, dt$$
$$= \int_0^1 F(t,t,t) \cdot (1,1,1)\, dt$$
$$= \int_0^1 (t^2, t^2, 1) \cdot (1,1,1)\, dt$$
$$= \int_0^1 (2t^2+1)\, dt = \frac{2t^3}{3} + t \Big|_0^1 = 5/3$$

11.

$$\int_{\gamma_2} F\,ds = \int_0^1 F(t, t^2, t^3) \cdot (1, 2t, 3t^2)\,dt$$

$$= \int_0^1 (t^4, t^4, 1) \cdot (1, 2t, 3t^2)\,dt$$

$$= \int_0^1 (t^4 + 2t^5 + 3t^2)\,dt = \frac{t^5}{5} + \frac{2t^6}{6} + t^3 \Big|_0^1$$

$$= \frac{1}{5} + \frac{2}{6} + 1 = 23/15.$$

$$\int_{\gamma_1} F\,ds = \frac{25}{15} \qquad \int_{\gamma_2} F\,ds = \frac{23}{15}.$$

Properties of the line integral.

(1) The line integral is independent of orientation preserving parametrizations of the curve.

i.e. Let $\gamma : [a, b] \longrightarrow \Omega \subset \mathbb{R}^n$ be a C^1 curve let $\theta : [c, d] \longrightarrow [a, b]$ a C^1 function such that $\theta(c) = a$, $\theta(d) = b$, with $\theta'(t) > 0$ $\forall t \in [c, d]$.

Then $\gamma \circ \theta : [c, d] \longrightarrow \Omega$ is a curve.

Then $\displaystyle\int_{\gamma\circ\theta} V\cdot ds = \int_c^d V((\gamma\circ\theta)(t)).(\gamma\circ\theta)'(t)\,dt$

$$= \int_c^d V(\gamma(\theta(t))).\gamma'(\theta(t))\,\theta'(t).dt$$

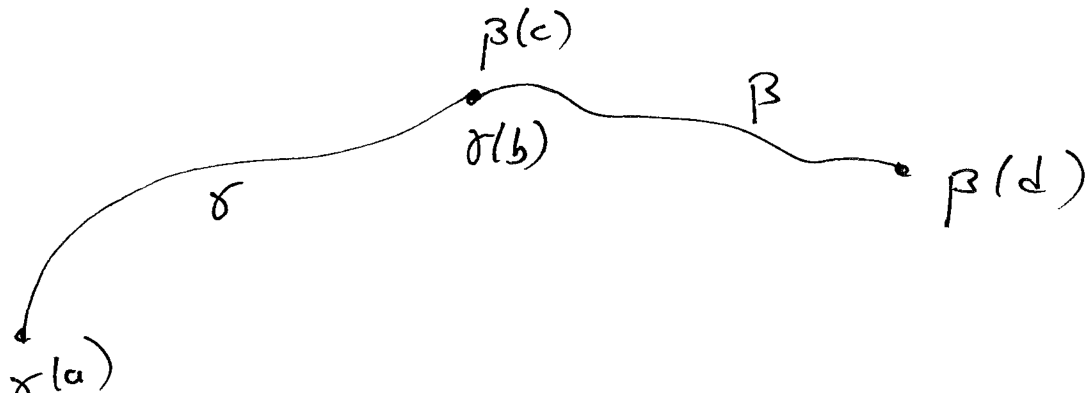
$$u=\theta(t) \quad = \int_a^b V(\gamma(u)).\gamma'(u)\,du = \int_\gamma V.ds.$$

Geometrically this means that $\int_\gamma V\,ds$ depends only on the image $(\gamma([a,b]))$ with the given orientation

(2) Let $\gamma : [a, b] \longrightarrow \Omega$.

$\beta : [c, d] \longrightarrow \Omega$

2 paths with $\gamma(b) = \beta(c)$



We define $\gamma + \beta$ the path formed by juxtposing the two paths γ, β.

i.e. $$\gamma + \beta := \begin{cases} \gamma(t) & t \in [a, b]. \\ \beta(t - b + c) & t \in [b, d + b - c]. \end{cases}$$

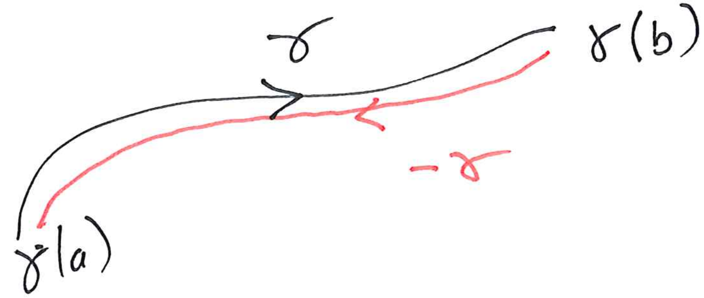
Then $\displaystyle\int_{\gamma+\beta} v \cdot ds = \int_{\gamma} v\, ds + \int_{\beta} v\, ds.$

③ Let $\gamma : [a,b] \to \Omega$ a path and

$-\gamma : [a,b] \to \Omega$ the same path traced in opposite direction

i.e. $(-\gamma)(t) := \gamma(a+b-t)$

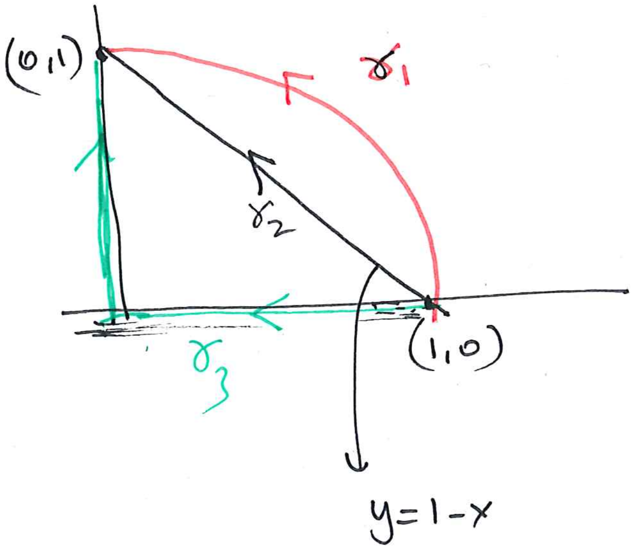
Then $\displaystyle\int_{-\gamma} v\, ds = -\int_{\gamma} v\, ds.$



In example 3 we've seen $v(x,y,z) = (y^2, xz, 1)$.

has different line integrals along 2 different curves between $(0,0,0)$ and $(1,1,1)$

Ex: $v(x,y) = \begin{pmatrix} y-x \\ x \end{pmatrix}$



- γ_1 = the curve along the unit circle.

$\gamma_1(t) = (\cos t, \sin t)$ $\quad t \in [0, \pi/2]$

$\gamma_2(t) = (1-t, t)$ $\quad t \in [0,1]$.

$\gamma_3(t) = c_1 + c_2$

$c_1(t) = (1-t, 0)$ $\quad t \in [0,1]$

$c_2(t) = (0, t)$ $\quad t \in [0,1]$.

$$\int_{\gamma_1} v\,ds = \int_0^{\pi/2} v(\gamma_1(t))\cdot\gamma_1'(t)\,dt = \int_0^{\pi/2} (\sin t - \cos t, \cos t)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^{\pi/2} -\sin^2 t - \cos t\sin t + \cos t\cos t\,dt = \dots \frac{1}{2}.$$

$$\int_{\gamma_2} v\,ds = \int_0^1 v(\gamma_2(t))\gamma_2'(t)\,dt = \int_0^1 (t-(1-t), 1-t)\cdot(-1,1)\,dt$$

$$= \int_0^1 (1-2t+1-t)\,dt = \int_0^1 2-3t\,dt = \frac{1}{2}$$

$$\int_{\gamma_3} v\,ds = \int_{c_1} v\,ds + \int_{c_2} v\,ds = \int_0^1 (t-1, 1-t)\cdot(-1,0)\,dt$$

$$+ \int_0^1 (t,0)\cdot(0,1)\,dt$$

$$= \frac{1}{2}.$$

It raises the question whether $\int_\gamma v\, ds$

is independent of the path between $(1,0)$ and $(0,1)$?

<u>Example.</u> Let $f : R^n \to \mathbb{R}$, $f \in C^1$

a scalar field, let $V = \nabla f : \mathbb{R}^n \to \mathbb{R}^n$, $x_0 \mapsto (\nabla f)(x_0)$.

be its gradient vector field.

Let $\gamma : [a,b] \to \Omega$ be a C^1 curve.

$$\int_\gamma V \cdot ds = \int_a^b \underbrace{\nabla f(\gamma(t)) \cdot \gamma'(t)}_{\frac{d}{dt}(f \circ \gamma)}\, dt = \int_a^b \frac{d}{dt}(f\circ\gamma)\, dt$$

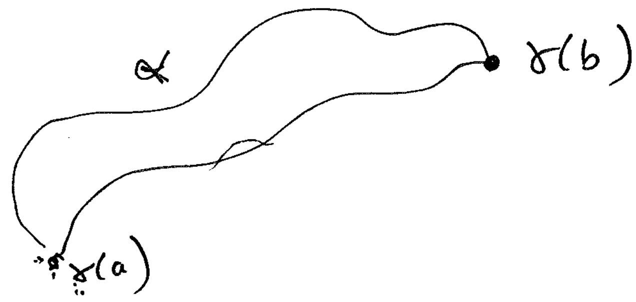
$$= (f\circ\gamma)(b) - (f\circ\gamma)(a)$$

1 dim'l fund. thm of integral analysis

$f \circ \gamma : [a,b] \subset \mathbb{R} \to \mathbb{R}$.

$$\int_\gamma v\, ds = f(\gamma(b)) - f(\gamma(a)).$$

$$= \int_\gamma \nabla f\, ds$$



$$\int_a^b F'\, dx = F(b) - F(a).$$

— o —

$v(x,y) = (y-x, x)$

Is there an $f: \mathbb{R}^2 \to \mathbb{R}$ such that

$$\nabla f = v$$

we need that $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \overset{?}{=} (y-x, x)$

y for example we take $f = yx - \frac{x^2}{2}$ then $\frac{\partial f}{\partial x} = y - x$, $\frac{\partial f}{\partial y} = x$

and $\nabla f = v$.

Defn. A differentiable scalar field $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = v$ is called a Potential for v

Remark For $n=1$, a potential is same as a primitive.

A function $G: \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$ is called a primitive of g if $G' = g$.
(stammfunktion)

Ex: For $v = (2xy^2, 2yx^2)$

$f(x,y) := f = x^2y^2$ is a potential. $\underbrace{\nabla f = (2xy^2, 2yx^2)}_{= v}$.

Recall In 1 dim given g, if ~~g~~ g is continuous then $G(x) := \int_a^x g(t)\,dt$ is a primitive of g.

For continuous $g : \mathbb{R} \to \mathbb{R}$, there is always a primitive.

Remark: For $n \geq 2$, there are many vector fields that do NOT have potentials.

Ex. If there were $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = v = (2xy^2, 2)$, then we should have had

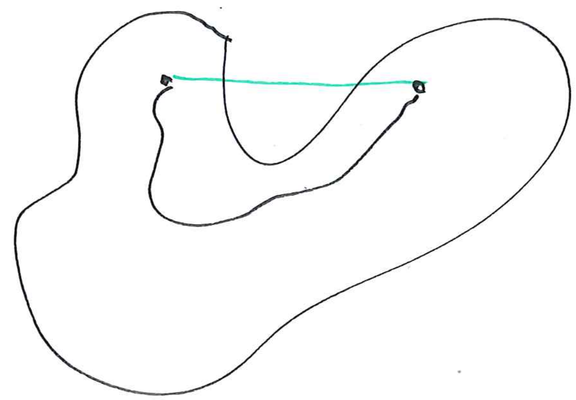
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xy^2, 2)$$

$$\Longrightarrow \quad \frac{\partial f}{\partial x} = 2xy^2 \quad \Longrightarrow \quad f = x^2 y^2 + c(y)$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = 2x^2 y + c'(y) = 2.$$ has no solution,

<u>Question</u>: When is a vector field $v: \mathbb{R}^n \to \mathbb{R}^n$ the gradient field of a scalar field $f: \mathbb{R}^n \to \mathbb{R}$

<u>We define</u>: <u>Defn</u> Let $\Omega \subset \mathbb{R}^n$ an open sset of $\mathbb{R}^n$. Ω is called (path) connected if for every pair of points $x, y \in \Omega$, $\exists$ a C^1 path $\gamma: [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$ such that $\gamma([0,1]) \subset \Omega$.



connected but not convex

Every convex set is connected.

<u>Thm</u> (Satz 7.4.2 in Struwe). Let v be a vector field which is continuous on an open connected set $\Omega \subset \mathbb{R}^n$. Then the following 3 statements are equivalent.

1) v is the gradient of some potential function f on Ω (ie $v = \nabla f$).

2) The line integral of v is independent of the path in Ω.

Ie ~~if~~ For each piecewise smooth C^1 path

$\gamma_i : [a_i, b_i] \to \Omega$ with some beginning $i = 1, 2.$

and end points ie. $\gamma_1(a_1) = \gamma_2(a_2)$, $\gamma_1(b_1) = \gamma_2(b_2)$.

we have $\int_{\gamma_1} v\, ds = \int_{\gamma_2} v\, ds$.

3) The line integral of v is zero around every piecewise smooth closed path in Ω.

if $\gamma : [a, b] \to \Omega$ with $\gamma(a) = \gamma(b)$

then $\int_\gamma v \cdot ds = 0$. $\oint_\gamma v\, ds$.