$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, e = unit vector.

The directional derivative of f at $x = x_0$ in the direction of e is

$$f'(x_0; e) := \lim_{h \to 0} \frac{f(x_0 + he) - f(x_0)}{h} \quad \text{(if the limit exists).}$$

$$= \frac{d}{dt} f(x_0 + et) \Big|_{t=0} .$$

For $e = e_i$, $f'(x_0, e_i)$ is also called the partial derivative of f in the direction e_i or simply i-th partial derivative and is denoted by $\frac{\partial f}{\partial x^i}(x_0)$ or $f_i(x_0)$ or $f_i'(x_0)$.

For functions of several variables

f has a directional derivative $\forall e$, unit vector in $x = x_0$ $\not\Rightarrow$ f is continuous at $x = x_0$.

$\underline{\text{Ex}}$: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

The stronger differentiability criteria is given by the "total derivative"

Defn The function $f : \mathbb{R}^n \to \mathbb{R}$ is in $x_0 \in \mathbb{R}^n$ (total) differentiable if $\exists$ a linear map $A = A_{x_0} : \mathbb{R}^n \to \mathbb{R}$ s.t

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - A_{x_0}(x - x_0)|}{\|x - x_0\|} = 0 .$$

If that is the case the linear transformation $A_{x_0} : \mathbb{R}^n \to \mathbb{R}$ is called the (total) derivative of f at x_0

It is denoted by $df(x_0)$ or $d_{x_0} f$

!!! Note ① the total derivative is NOT a number, it is a linear map

② If df exists then it is unique.

$f: \mathbb{R}^n \to \mathbb{R}$ differentiable at $x = x_0$, with total derivative $d_{x_0} f$.
$d_{x_0} f : \mathbb{R}^n \to \mathbb{R}$ a linear map.

Then **Thm** If f as above then the derivative of f at x_0 with respect to any $y \in \mathbb{R}^n$ exists and we have

$$f'(x_0 ; y) = d_{x_0} f(y).$$

In particular $\frac{\partial f}{\partial x^i}(x_0) = d_{x_0} f(e_i)$

The Matrix representation of $d_{x_0} f : \mathbb{R}^n \to \mathbb{R}$ with respect to the Basis $\{e_1, \dots, e_n\}$ is the Matrix

$$\nabla f(x_0) := \left(\frac{\partial f}{\partial x^1}(x_0), \frac{\partial f}{\partial x^2}(x_0), \dots, \frac{\partial f}{\partial x^n}(x_0) \right)$$

and $f'(x_0 ; y) = \langle \nabla f(x_0), y \rangle$.

$\nabla f = \left(\frac{\partial f}{\partial x^1}, \dots \frac{\partial f}{\partial x^n} \right)$ is called the gradient of f

Thm f is total differentiable in $x_0 \Longrightarrow f$ is cont. in x_0.

$$\nabla f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \text{is a vector field.}$$
$$x_0 \longrightarrow \nabla f(x_0)$$

First order Taylor Formula for f. $f : \mathbb{R}^n \to \mathbb{R}$ diff in x_0. Then.

$$\boxed{f(\bar{x}) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + R(x, x_0)}$$

with $\displaystyle \lim_{x \to x_0} \frac{R(x, x_0)}{|x - x_0|} = 0$.

e.g. $f : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to f(x,y)$

$$f(x,y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0) + R((x,y), x_0, y_0)).$$

$$z = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)(y - y_0)$$

$$\underline{z = z_0 + A_1(x - x_0) + A_2(x - x_0)};$$ with

$$z_0 = f(x_0, y_0)$$
$$A_1 = \frac{\partial f}{\partial x}(x_0, y_0)$$
$$A_2 = \frac{\partial f}{\partial y}(x_0, y_0)$$

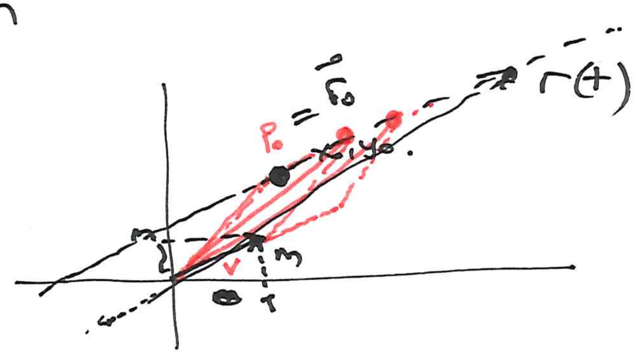
Digression – Review of lines and planes in space.

Lines. $\mathbb{R}^2$ In plane a line is determined by a point P_0 on the line and a # giving the slope of the line.

General eqn of a line in $\mathbb{R}^2$ $\boxed{y = mx + b.}$

if $P_0 = (x_0, y_0)$ then

$$y - y_0 = m(x - x_0).$$



We can also write this eqn.

as $(x,y) = \underbrace{(x_0, y_0)}_{P_0} + \underbrace{(1,m)}_{\vec{v}}t, \quad t \in \mathbb{R}.$

$\boxed{(x,y) = P_0 + \vec{v}t}$ $t \in \mathbb{R}.$ vektor eqn of the line.

$\vec{r}(t) = \vec{r}_0 + \vec{v}\,t$

$x = x_0 + t \quad , \quad y = y_0 + mt \qquad t \in \mathbb{R}.$ parametric eqn of the line

We can also write this as

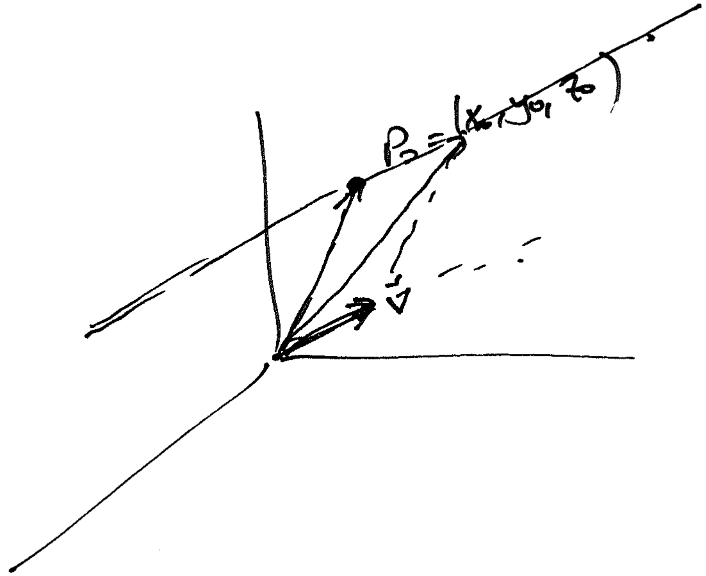
$$\boxed{\frac{x - x_0}{1} = \frac{y - y_0}{m}.}$$

$\mathbb{R}^3$ **Eqn of a line in space**

Any line is determined uniquely by a point P_0 and a vector giving the direction of the line.

Be careful!!! In space the eqn $y = mx + b$ is NOT a line. This is a plane in $\mathbb{R}^3$.

In $\mathbb{R}^3$ "slope" is given by a vector of direction. $\vec{v}$



$(x, y, z) = P_0 + \vec{v}t \quad t \in \mathbb{R}.$

In $\mathbb{R}^n$ we can write eqn of a "line"

as $P_0 + \vec{v}t \quad , t \in \mathbb{R}.$

$$\boxed{\vec{r}(t) = P_0 + \vec{v}t} \qquad v = (v_1, v_2, v_3)$$

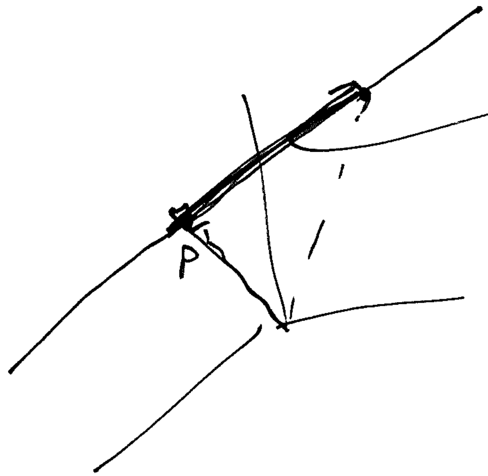
$$\boxed{\begin{aligned} x &= x_0 + v_1 t \\ y &= y_0 + v_2 t \\ z &= z_0 + v_3 t \end{aligned}} \quad , \; t \in \mathbb{R}.$$

$$\boxed{\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}}$$

Parametrizing a line segment on a line:

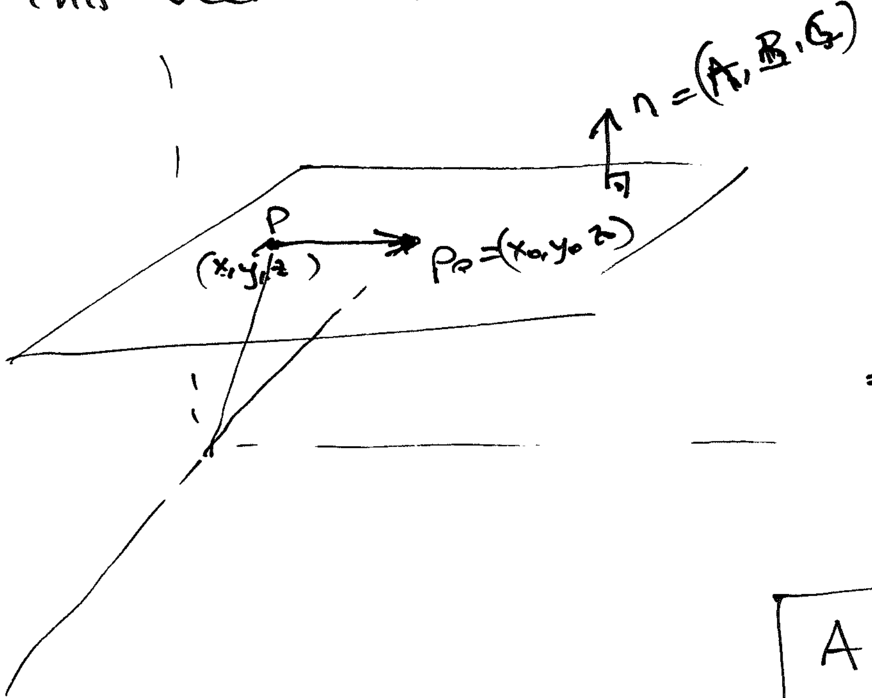
from a point $\vec{P}$ to Q on the line.

Then it is given $\vec{r}(t) = \vec{P}(1-t) + \vec{Q}t \qquad t \in [0,1]$



Eqns of Plane in $\mathbb{R}^3$.

A plane in $\mathbb{R}^3$ is determined uniquely by a point $P_0 \in \mathbb{R}^3$ and by its "tilt" which is determined by a vector which is orthogonal to the plane. This vector is called the normal vector. If P is any other point on the plane.



$\overrightarrow{PP_0}$ is on the plane

$\overrightarrow{P_0P} \perp \vec{n}$

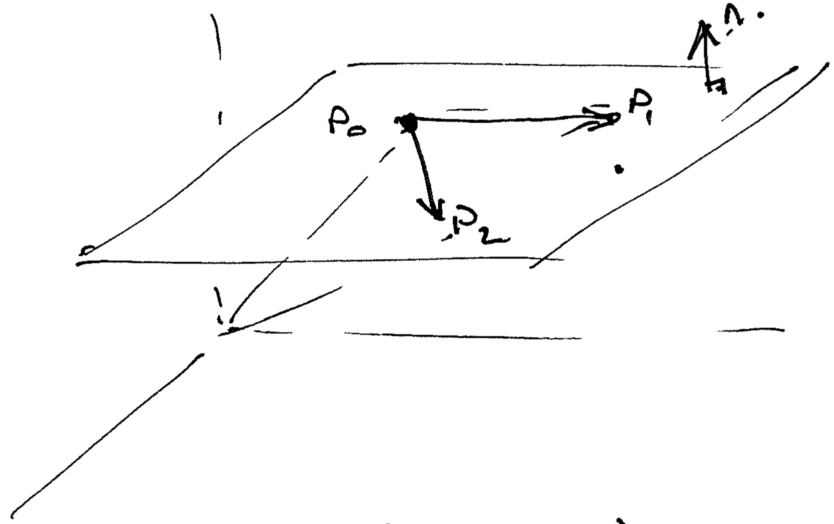
$$\boxed{(x-x_0, y-y_0, z-z_0) \perp (A, B, C)}$$

$$\Rightarrow \boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0}$$

$$Ax + By + Cz = \underbrace{z_0 + Ax_0 + By_0}_{=:D}$$

$$\boxed{Ax + By + Cz = D.}$$

A plane is also determined by 3 points that do not all lie on a line.



$$P = P_0 + P_1P_0\, t + P_2P_0\, s.$$
$$s, t \in \mathbb{R}.$$

Ex.

$P_1 = (1, -2, 0)$
$P_2 = (3, 1, 4)$
$P_3 = (0, -2, 4)$

$P_1P_2 = (2, 3, 4)$
$P_1P_3 = (-1, 0, 4)$.

$$(x, y, z) = (1, -2, 0) + (2, 3, 4)t + (-1, 0, 4)s \quad , \quad s, t \in \mathbb{R}.$$

$$\left.\begin{array}{l} x = 1 + 2t - s \\ y = -2 + 3t \\ z = 4t + 4s. \end{array}\right\}.$$

To find the scalar eqn of the same plane,
we need to find a normal vector, n

$n \perp P_1P_2$ and $n \perp P_1P_3$ $\Rightarrow$ n can be found by the ~~taking~~ the cross product of the 2 vectors P_1P_2, P_1P_3

$$\underset{P_1P_2}{(2,3,4)} \times \underset{P_1P_3}{(-1,0,4)} = (12,-12,3)$$

$$\det \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ -1 & 0 & 4 \end{vmatrix} = i\,12 - 12j + 3k. \Rightarrow \underline{(12,-12,3)} = \vec{n}.$$

Take $P_1 = (-1, 2, 0)$ $\vec{n} = (12,-12,3)$

eqn of the plane $(x+1, y-2, z) \cdot (12,-12,3) = 0.$

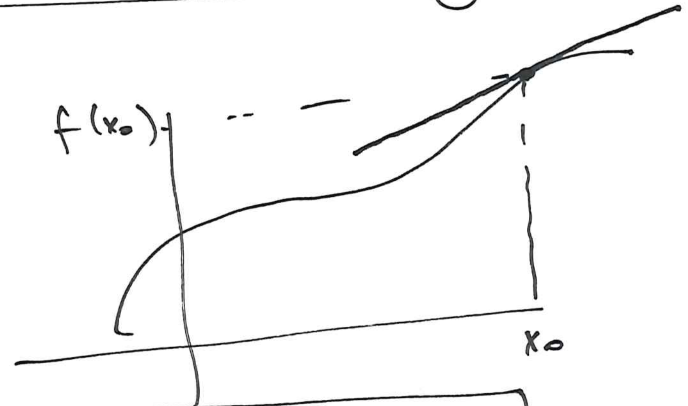
$$12(x+1) - 12(y-2) + 3z = 0.$$

$$\boxed{12x - 12y + 3z = -36}.$$

$$\begin{cases} y = mx + b \\ mx - y + 0 \cdot z = b. \end{cases} \rightarrow \text{is a plane}$$

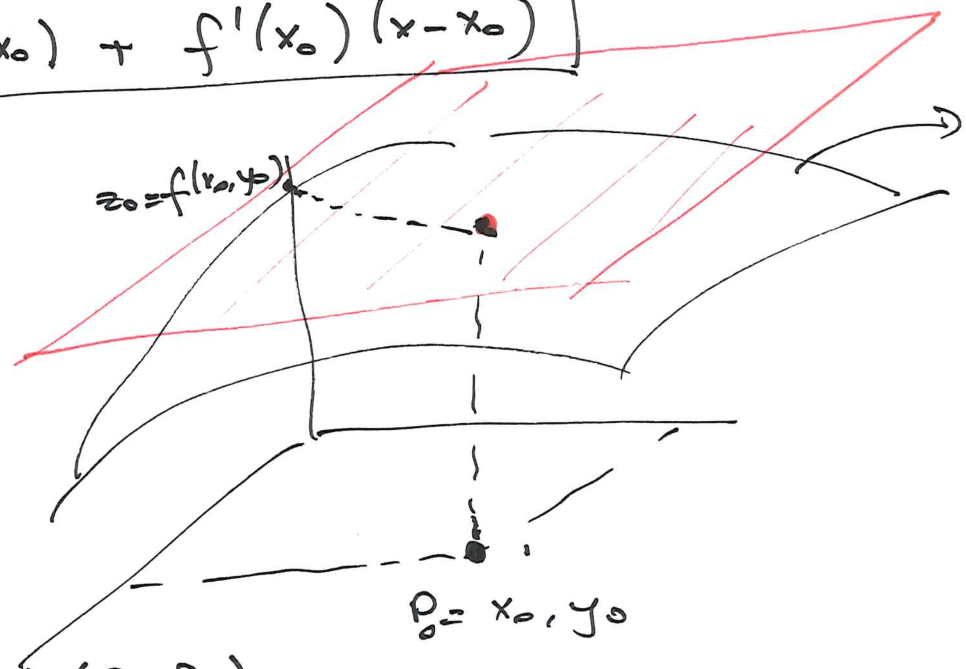
The differential and the tangent plane.

$f : \mathbb{R} \to \mathbb{R}.$



Tangent line : $\boxed{y = f(x_0) + f'(x_0)(x - x_0)}$

$f : \mathbb{R}^2 \to \mathbb{R}$



$$f(P) \sim \underbrace{f(P_0)}_{z_0} + \nabla f(P_0) \cdot (P - P_0)$$

$$\left(\frac{\partial f}{\partial x}(P_0), \frac{\partial f}{\partial y}(P_0) \right) \cdot (x - x_0, y - y_0).$$

$$f(P) \sim \boxed{z_0 + \underbrace{\frac{\partial f}{\partial x}(P_0)}_{A}(x - x_0) + \underbrace{\frac{\partial f}{\partial y}}_{B}(y - y_0).}$$

$$z = z_0 + A(x - x_0) + B(y - y_0)$$

$$0 = A(x - x_0) + B(y - y_0) - (z - z_0)$$

$$0 = (A, B, -1) \cdot (x - x_0, y - y_0, z - z_0)$$

Ex. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x, y) \longrightarrow \sqrt{x^2 + y^2}$.

$(x_0, y_0) = (3, 4)$. Find the eqn of the tangent plane to f at $(x_0, y_0) = (3, 4)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

$$\nabla f(3,4) = \left(\frac{3}{5}, \frac{4}{5} \right) \qquad f(3,4) = 5$$

The tangent plane: $z = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$.

$$z = 5 + \left(\frac{3}{5}, \frac{4}{5} \right) \cdot (x - 3, y - 4)$$

$$\boxed{z = 5 + \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4)}.$$

← GeoGebra

Tangent plane and normal line to a surface

This applet illustrates the computation of the normal line and the tangent plane to a surface at a point P.

1. Select the point $P = (x_0, y_0)$ where to compute the normal line and the tangent plane to the graph of f using the sliders.
2. Check the box Normal line to plot the normal line to the graph of f at the point P, and to show its equation.
3. Check the box Tangent plane to plot the tangent plane to the graph of f at the point P, and to show its equation.
4. Enter a new function and repeat the previous steps.

Funtion: (x² + y²)^(1 / 2)

Point P=(3,4)

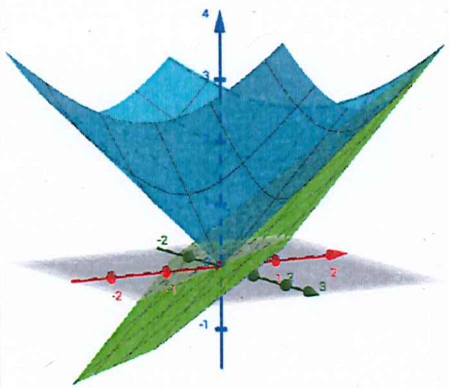
$x_0 = 3$

$y_0 = 4$

☐ Normal line

☑ Tangent plane

$\Pi : (x - 3, y - 4, z - 5)(-0.6, -0.8, 1) =$ 0



www.geogebra.org,

Get more info about the normal line and the tangent plane to a surface at http://aprendeconalf.es.

← GeoGebra

Tangent plane and normal line to a surface

This applet illustrates the computation of the normal line and the tangent plane to a surface at a point P.

1. Select the point $P=(x_0, y_0)$ where to compute the normal line and the tangent plane to the graph of f using the sliders.
2. Check the box Normal line to plot the normal line to the graph of f at the point P, and to show its equation.
3. Check the box Tangent plane to plot the tangent plane to the graph of f at the point P, and to show its equation.
4. Enter a new function and repeat the previous steps.

Funtion: x² + y²

Point P=(1,1)

$x_0 = 1$

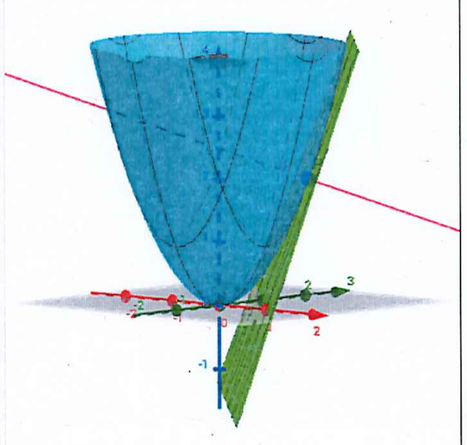
$y_0 = 1$

☑ Normal line

$l : (1, 1, 2) + t(-2, -2, 1)$

☑ Tangent plane

$\Pi : (x-1, y-1, z-2)(-2, -2, 1) = 0$

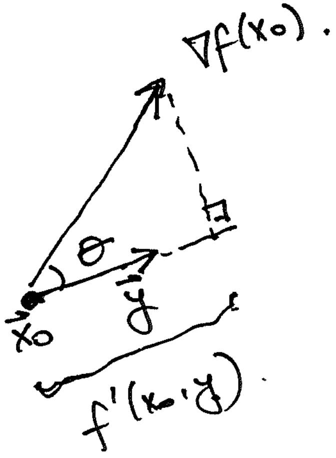


Get more info about the normal line and the tangent plane to a surface at http://aprendeconalf.es.

www.geogebra.org

We've seen that $f'(x_0, y) = \langle \nabla f(x_0), y \rangle$.

$$= \|\nabla f(x_0)\| \, \|y\| \cos\theta$$



If y is a unit vector

$$f'(x_0, y) = \|\nabla f(x_0)\| \cos\theta.$$

That means $f'(x_0, y)$ is the component of $\nabla f(x_0)$ in the direction of y.

This derivative is largest if $\cos\theta = 1. \Rightarrow \theta = 0.$

i.e. y has the same direction as ∇f.

i.e. f increases fastest in the direction of $\nabla f(x_0)$.

__Remark__. $f : \mathbb{R}^n \to \mathbb{R}$ differentiable.

$df(x_0)$ has matrix repn $\left(\frac{\partial f}{\partial x^1}(x_0), \ldots \frac{\partial f}{\partial x^n}(x_0) \right)$

The map $x^i : \mathbb{R}^n \to \mathbb{R}$
$x \to x^i$ is diff. at every pt x_0.

$dx^i = (0, 0 \ldots 1, 0 \ldots 0)$.

The differentials $dx^1, dx^2, \ldots dx^n$ form a basis for the linear space

$$\mathcal{L}(\mathbb{R}^n \to \mathbb{R}) = \{ A \mid A : \mathbb{R}^n \to \mathbb{R}, A \text{ linear} \}.$$

where we identify $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ with matrix repn
$A = (A_1 \ldots A_n)$ $A_i = A(e_i)$.

$$A = \boxed{A_1 dx^1 + A_2 dx^2 + \ldots + A_n dx^n.}$$

$$\boxed{df = \frac{\partial f}{\partial x^1}(x_0) dx^1 + \frac{\partial f}{\partial x^2}(x_0) dx^2 + \ldots + \frac{\partial f}{\partial x^n}(x_0) dx^n.}$$

examples of __1-forms.__

A sufficient condition for differentiability.

We've seen: f is diff at x_0 $\Longrightarrow$ $\frac{\partial f}{\partial x^i}(x_0)$ exist

$f: \mathbb{R}^n \to \mathbb{R}$.

f is diff at x_0 $\not\Longleftarrow$ $\frac{\partial f}{\partial x^i}(x_0)$ exist

Thm. (7.1.1 Shme). ~~A~~ssume $\frac{\partial f}{\partial x^i}$ $i=1 \dots n$ exist in a neighborhood of x_0 and continuous at x_0

Then f is differentiable at x_0.

<u>Summarize:</u>

To study diff. of $f : \mathbb{R}^n \to \mathbb{R}$.

1) f is diff in x_0 $\Longrightarrow$ $\frac{\partial f}{\partial x^i}(x_0)$ exist

$$df(x_0) = \nabla f(x_0).$$

2) $f \in C^1$ $\left\{ \frac{\partial f}{\partial x^i} \text{ exist and continuous} \right\}$ $\Longrightarrow$ f is differentiable.

$\boxed{\text{Is } f \text{ diff at } x_0\text{?}}$

Is f continuous at x_0?

- No $\to$ f is not diff.
- Yes $\to$ Is f in x_0 partial differentiable? i.e. $\frac{\partial f}{\partial x^i}$ exist
  - No $\to$ f is not diff.
  - Yes $\to$ Are $\frac{\partial f}{\partial x^i}$ continuous.
    - No. $\to$ *
    - Yes $\to$ f is diff

(*) . Is there a linear map $A: \mathbb{R}^n \to \mathbb{R}$ such that , its matrix rep is $\nabla f(x_0)$

$$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)}{\|x - x_0\|} \right| = 0 .$$

If yes it is differentiable if ~~not~~ it is not.

Differentiation rules.

<u>Thm</u> Let $\Omega \subset \mathbb{R}^n$, $f, g : \Omega \to \mathbb{R}$ such that f, g are differentiable at x_0.

Then $f+g$, $f \cdot g$ are diff in x_0 and we have.

1) $d(f+g)(x_0) = df(x_0) + df(g_0)$.

2) $d(fg)(x_0) = (df)(x_0) \cdot g(x_0) + f(x_0) \cdot (dg(x_0))$

3) if $g(x_0) \neq 0$ then f/g is also differentiable

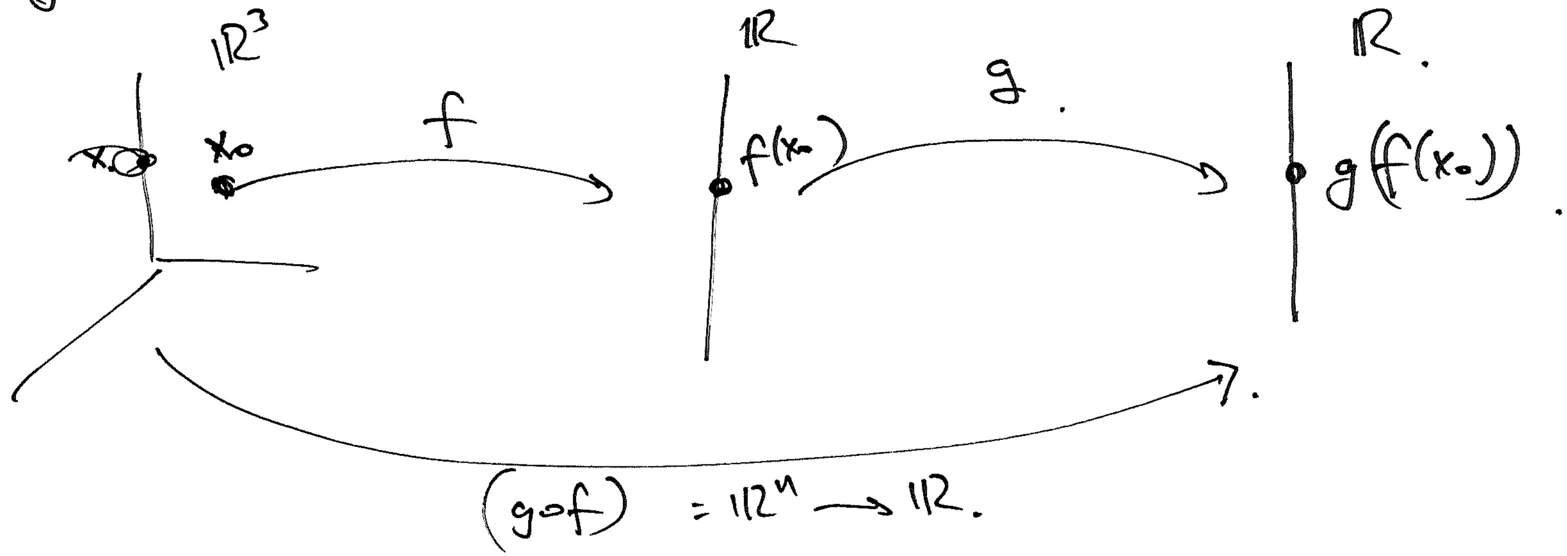
$$d\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0) \cdot df(x_0) - f(x_0)(dg(x_0))}{g(x_0)^2}$$

__Thm__ (Chain rule - Version 1).

Let $f : \mathbb{R}^n \to \mathbb{R}$ diff in x_0, and
$g : \mathbb{R} \to \mathbb{R}$ diff in $f(x_0)$

Then $g \circ f : \mathbb{R}^n \to \mathbb{R}$ is diff in x_0 and

$$d(g \circ f)(x_0) = g'(f(x_0))\, df(x_0).$$



Example : $h: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to e^{xy}$

$h = g \circ f$ where $f(x,y) = xy$, $g(t) = e^t$

$f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to xy$.

$g: \mathbb{R} \to \mathbb{R}$, $t \to e^t$

We can find ~~df~~ dh directly

$$dh(x_0,y_0) = \nabla h(x_0,y_0) = \left(\frac{\partial h(x_0,y_0)}{\partial x}, \frac{\partial h(x_0,y_0)}{\partial y}\right)$$

$$\boxed{dh = (ye^{xy}, xe^{xy}).}$$

or use the chain rule

$$dh(x,y) = d(g \circ f) = g'(f(x,y)) \cdot (df(x,y))$$

$$= e^{xy}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = e^{xy}(y, x)$$

$$\boxed{dh = (e^{xy}y, e^{xy}x)}$$

For the 2nd chain rule, we first give a defn.

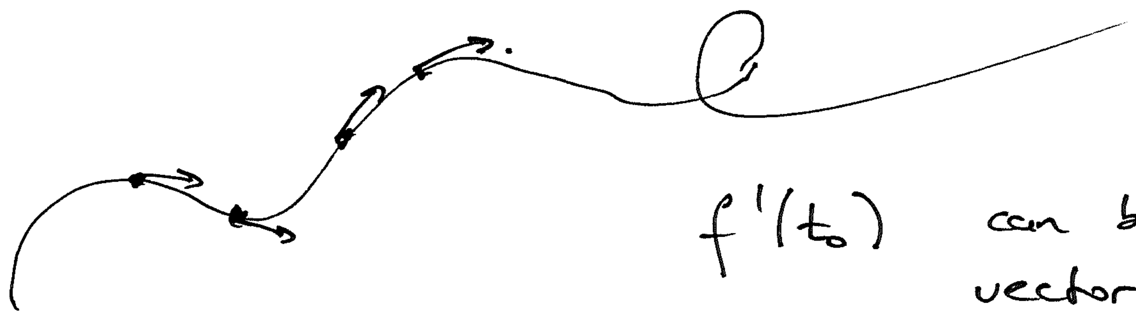
<u>Defn.</u> Let $I \subseteq \mathbb{R}$, $f \cancel{(f_1, f_2, \dots, f_n)} : I \longrightarrow \mathbb{R}^n$, $t \longmapsto (f_1(t), f_2(t), \dots, f_n(t))$

we say f is differentiable in $x_0 \in I$

if each component function $f_i : I \longrightarrow \mathbb{R}$ is diff in x_0.

In this case $f'(t) := (f_1'(t), f_2'(t), \dots, f_n'(t))$.

<u>Rk.</u>



$f : I \to \mathbb{R}^n$ describes a curve in $\mathbb{R}^n$.

$f'(t_0)$ can be seen as a velocity vector at the point $f(t_0)$

<u>Thm</u> (Chain rule - version II)

Let $\Omega \subset \mathbb{R}^n$, $I \subset \mathbb{R}$ $\quad f : \Omega \longrightarrow \mathbb{R}$

$g : I \longrightarrow \Omega \subset \mathbb{R}^n$

Assume g is differentiable at $t_0 \in I$ and f is differentiable at $g(t_0)$.

Then the function $f \circ g : I \longrightarrow \mathbb{R}$ is differentiable in t_0 and.

$$\boxed{\frac{d}{dt}(f \circ g) = (df)(g(t_0)) \cdot g'(t_0)}.$$

$$\left\langle \left(\frac{\partial f}{\partial x^1}(g(x_0)), \dots \frac{\partial f}{\partial x^n}, g(x_0) \right), \left(g_1'(t_0), g_2'(t_0), \dots g_n'(t_0) \right) \right\rangle.$$

$$\frac{\partial f}{\partial x^1}(g(x_0))\, g_1'(t_0) + \frac{\partial f}{\partial x^2}(g(x_0))\, g_2'(t_0) + \dots + \frac{\partial f}{\partial x^n}(g(x_0)) \cdot g_n'(t_0).$$

__Ex__. Basic operations of addition and multiplication are differentiable functions of 2 variables.

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \mapsto x+y.$$

$$M : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \mapsto xy.$$

$$dA = \left(\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}\right) = (1, 1).$$

$$dM = \left(\frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}\right) = (y, x)$$

Now we use these functions in the chain rule with

$$g : \mathbb{R} \to \mathbb{R}^2$$
$$t \to (g_1(t), g_2(t))$$

$g_i(t) : \mathbb{R} \to \mathbb{R}.$

$A \circ g(t) = g_1(t) + g_2(t).$ | $M \circ g(t) = g_1(t) g_2(t).$ >

$$d(A \circ g)(t) \underset{\substack{\downarrow \\ \text{chain rule}}}{=} (dA)(g(t)) \cdot g'(t)$$

$$\| \qquad (1,1) \cdot (g_1'(t), g_2'(t)).$$

$$= g_1'(t) + g_2'(t)$$

$$d(g_1(t) + g_2(t)) = g_1'(t) + g_2'(t)$$

$$d(M \circ g)(t) = (dM)(g(t)) \cdot g'(t)$$

$$= \langle (dM)(g_1(t), g_2(t)), (g_1'(t), g_2'(t)) \rangle$$

$$= \langle (g_2(t), g_1(t)), (g_1'(t), g_2'(t)) \rangle$$

$$= g_2(t) g_1'(t) + g_1(t) g_2'(t).$$

$$= d(g_1 g_2)(t)$$

Ex: If $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \to \sin(xyz)$ and $g : \mathbb{R} \to \mathbb{R}^3$, $t \to (t^2, t^3, t)$

Directly

$$(f \circ g)(t) = f(t^2, t^3, t) = \sin t^6$$

$$\frac{d(f \circ g)}{dt} = (\cos t^6)(6t^5).$$

or

using chain rule. $\frac{d}{dt}(f \circ g) = df(g(t)) \cdot g'(t)$

$$df = (yz\cos(xyz),\ xz\cos xyz,\ xy\cos xyz)$$

$$g'(t) = (2t, 3t^2, 1)$$

$$df(g(t)) \cdot (2t, 3t^2, 1) = 2t^5 \cos t^6 + 3t^5 \cos t^6 + t^5 \cos t^6$$
$$= \underline{6t^5 \cos t^6}$$