Thm (Green). Let $f(x,y) = (P(x,y), Q(x,y))$ be a vector field that is continuously differentiable on an open simply connected set $\Omega \subset \mathbb{R}^2$. Let γ be a piecewise smooth closed curve and let R be the union of γ and its interior. Then we have

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_\gamma P\,dx + Q\,dy.$$

where the line integral is taken around γ in the ccw direction

---

Note Since Ω is simply connected, $R \subset \Omega$,

Application of Green's theorem.

If γ is a simple closed curve which encloses a region Ω

$$\left.\begin{aligned} \text{area}(\Omega) &= \frac{1}{2}\oint_\gamma x\,dy - y\,dx \\ &= \oint_\gamma -y\,dx \\ &= \oint_\gamma x\,dy. \end{aligned}\right\}$$

Because in each case.

If $V = \frac{1}{2}(-y, x)$ or $(-y, 0)$ or $(0, x)$

then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} =: \text{rot}(V)$ is equal to 1.

In each case the line integral $= \iint_\Omega 1\,dx\,dy = \text{Area}(\Omega)$.

Ex 1 We can use this to find the area of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(we did this already using the double integral and we do know it is πab.

Now using Green's thm. γ: has parametrization $(a\cos t, b\sin t)$, $t\in[0,2\pi]$)

$\gamma'(t) = (-a\sin t, b\cos t)$

$$\oint_\gamma \frac{-y}{2}dx + \frac{x}{2}dy$$

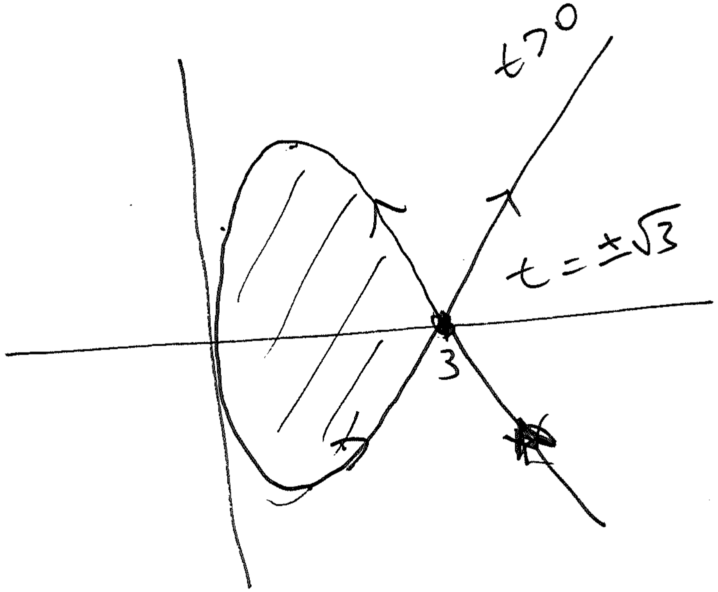
$$\int_0^{2\pi}\left(\frac{-b\sin t}{2}, \frac{a}{2}\cos t\right)\cdot(-a\sin t, b\cos t)\,dt = \frac{ab}{2}\int_0^{2\pi}(\underbrace{\sin^2 t+\cos^2 t}_{1})\,dt$$

$$= 2\pi\cdot\frac{ab}{2} = \pi ab,$$

Ex: Find the area enclosed by the parametric curve

$$\gamma(t) = \begin{pmatrix} t^2 \\ t^3/3 - t \end{pmatrix} \qquad -\sqrt{3} \leq t \leq \sqrt{3}$$

$$y^2 = ax^3 + \cdots$$



$$\text{Area} = \frac{1}{2} \oint_\gamma x\,dy - y\,dx$$

$$= \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \left(-\frac{t^3}{3} + t,\ t^2\right) \cdot (2t,\ t^2 - 1)\, dt$$

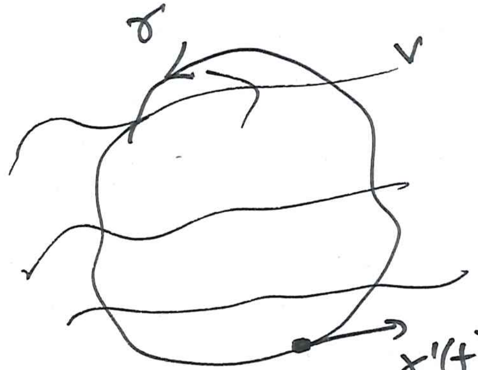
$$= \frac{1}{2} \int_{-\sqrt{3}}^{\sqrt{3}} -\frac{2}{3} t^4 + 2t^2 + t^4 - t^2 \; dt$$

$$= \cdots\cdots = \frac{8\sqrt{3}}{5}.$$

Line integrals as flow integrals

Let v represent the velocity field of a fluid flowing through a region.

The integral of $v \cdot T$ along γ in this region gives the fluid's flow along the curve



$\gamma'(t) = T$

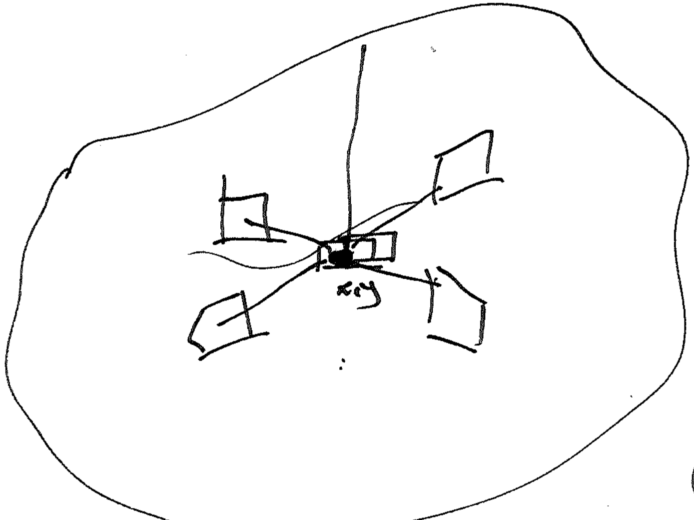
<u>Defn:</u> If $\gamma(t)$ is a smooth closed curve in the domain of a continuous velocity field v.

Then the <u>flow</u> along the curve γ from $t=a$ to $t=b$ is

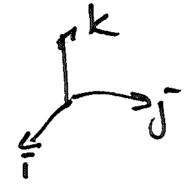
$$\text{Flow} = \int_a^b v \, d\gamma = \int_a^b v(\gamma(t)) \cdot \gamma'(t) \, dt = \int_\gamma v \cdot T \, ds$$

And the integral is called the flow integral.
If γ is a closed curve, it is called the circulation around the curve

The idea of Green's thm is motivated by the question of measuring how a paddle wheel (or a boat) spins at a point in the fluid flowing in a region with $v = (P, Q)$
There is something that is called the <u>curl</u> <u>rotation</u> or <u>circulation density</u>. We want to know how the fluid is circulating at different points around the axes perpendicular to the region.

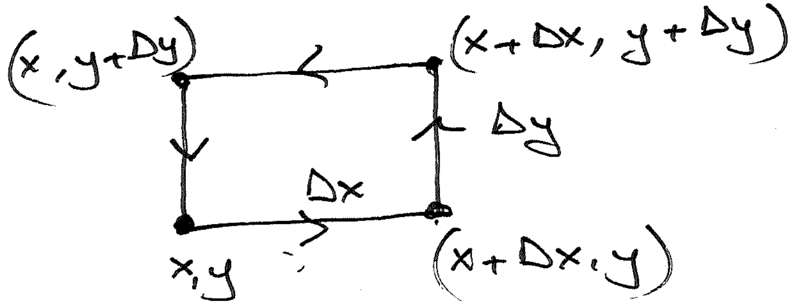


let $V = P(x,y)\,i + Q(x,y)\,j$



be the velocity field

Consider a small rectangle with one corner at this point



The ccw circulation of v around the boundary of this rectangle is the sum of flows along the 4 sides

At the bottom edge $\quad \cancel{V}\cdot i\,\Delta x \approx P(x,y)\,\Delta x$

top edge $\quad v(x, y+\Delta y)\cdot(-i\,\Delta x) = -P(x, y+\Delta y)\,\Delta x$

right $\quad \cancel{V}(x+\Delta x, y)\cdot j\,\Delta y = Q(x+\Delta x, y)\,\Delta y$

left $\quad \cancel{V}(x,y)\cdot -j\,\Delta y = -Q(x,y)\,\Delta y.$

7

If we add opposite pairs to gether

Top + bottom : $-\left(P(x, y+\Delta y) - P(x,y)\right) \Delta x$

right + left = $\left(Q(x+\Delta x, y) - Q(x,y)\right) \Delta y$

Top + bottom $\approx -\left(\frac{\partial P}{\partial y} \cdot \Delta y\right) \Delta x$

right + left $\approx \left(\frac{\partial Q}{\partial x} \cdot \Delta x\right) \Delta y.$

Adding these together and by dividing $\Delta x \Delta y$.

gives an estimate of the "circulation density"

$$\frac{\text{circulation around rectangle}}{\text{area of rectangle}} \approx \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{circulation density}}.$$

and it is denoted by rot(v).

in english it is called curl

Green's thm in the form. $v = (P, Q)$

$$\oint_\gamma v \cdot T \, ds = \underbrace{\oint_\gamma P\,dx + Q\,dy}_{\text{circulation}} = \underbrace{\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy}_{\text{integ. of circ. density}}$$

is called the tangential form or circulation - curl form.

<u>Remark.</u> If we view the 2-dim'l vector field $v = (P, Q)$ in space (i.e. in 3 dimensions) what we found as rotation is actually the z-component of a more general rotation which is in general vector.

For a general vector field F in $\mathbb{R}^3$ $F = (P, Q, R)$

$$\operatorname{curl} F = \operatorname{rot} F := \nabla \times F = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k.$$

There is a second form of Green's theorem which is equivalent to the first. It is called the <u>normal form</u> or <u>Flux-divergence</u> form

It is given as follows.

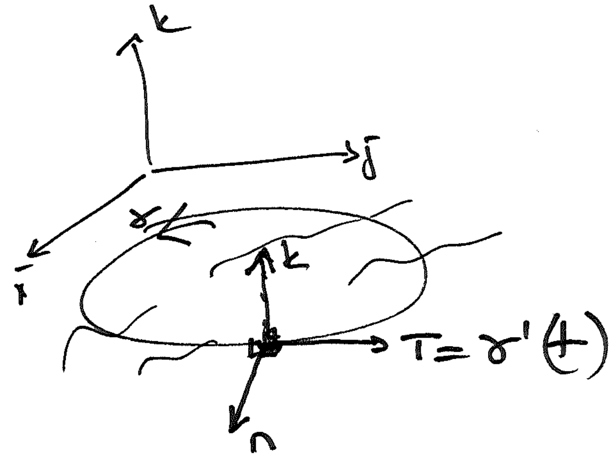
<u>Thm</u> (Green)

$$\boxed{\oint_\gamma P\,dy - Q\,dx = \iint_\Omega \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}_{\text{divergence of } V.} dx\,dy}$$

<u>Remark</u> (1) This is same as before if we take the vector field $F = (-Q, P)$

Why is this called the normal form?

If $v = (P, Q)$



$$\gamma(t) = (x(t),\ y(t)).$$

$$n = T \times k = (\gamma'(t) \times k)$$

$$\left(\frac{dx}{dt} i + \frac{dy}{dt} j\right) \times k.$$

$$= \frac{dy}{dt} i - \frac{dx}{dt} j$$

$$F \cdot n = (P, Q) \cdot \left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$$

$$\oint_\gamma (F \cdot n)\, ds = \oint_\gamma P\,dy - Q\,dx$$

$$\int_a^b \left(P(x,y)\frac{dy}{dt} - Q(x,y)\frac{dx}{dt}\right) dt$$

$$= \oint P\,dy - Q\,dx$$

In general for a vector field $F = (P, Q, R)$

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (P, Q, R)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ is called

the divergence of F, which is a scalar valued function.

Green's theorem has generalizations to 3 dimensions

* Normal Form :

$$\oint_{\gamma} F.n \, ds = \iint_{\Omega} \underbrace{\nabla . F}_{\text{divergence.}} \, dA \quad \text{in } \mathbb{R}^2$$

Divergence Theorem.

$$\underbrace{\iint_S F.n \, d\sigma.}_{\text{surface integral.}} = \iiint_D (\nabla . F) \, dV \quad \text{in } \mathbb{R}^3.$$

* Tangential Form.

$$\oint_{\gamma} F.T \, ds = \iint_{\Omega} \overbrace{(\nabla \times F)}^{\text{rotation}} . k \, dA$$

Stoke's theorem

$$\oint_{\gamma} F.T \, ds = \iint_S (\nabla \times F) \bullet n \, d\sigma.$$

§. Change of variables in multiple integrals.

n=1 Change of variables – substitution formula.

Let $U = \text{interval} = [a, b] \subset \mathbb{R}$

Let $\varphi : U \longrightarrow \mathbb{R}$ which is injective

let $X = \varphi(U)$ then $\varphi : U \longrightarrow \varphi(U) = X$ is a bijection

such that $\varphi' \neq 0$ on U.

If it is increasing then $\underline{X} = \varphi([a,b]) \underset{=[c,d]}{} = (\varphi(a), \varphi(b))$

if it is decreasing then $\underline{X} = (\varphi(b), \varphi(a))$.

$$\int_a^b f(\varphi(u))\,\varphi'(u)\,du. \qquad \text{Now put} \quad \begin{aligned} x &= \varphi(u) \\ dx &= \varphi'(u)\,du. \end{aligned}$$

$$= \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx = \begin{cases} \displaystyle\int_c^d f(x)\,dx & \varphi' > 0 \\ = \displaystyle\int_d^c f(x)\,dx. & \varphi' < 0. \end{cases}$$

Hence

$$\int_{U=[a,b]} f\,\varphi(u)\,\varphi'(u)\,du = \operatorname{sgn}\varphi' \int_{\varphi(u)=X} f(x)\,dx.$$

~~Namely~~ we have

Thm (8.5.2) Let U be an open set in $\mathbb{R}^n$

$\varphi : U \longrightarrow \mathbb{R}^n$ injective C^1 map with

Jacobian matrix $\nabla\varphi$.

Assume that $\det(\nabla\varphi)(u) \neq 0 \quad \forall u \in U$.

Let $X = \varphi(U)$ and $f : X \to \mathbb{R}$ a continuous scalar field

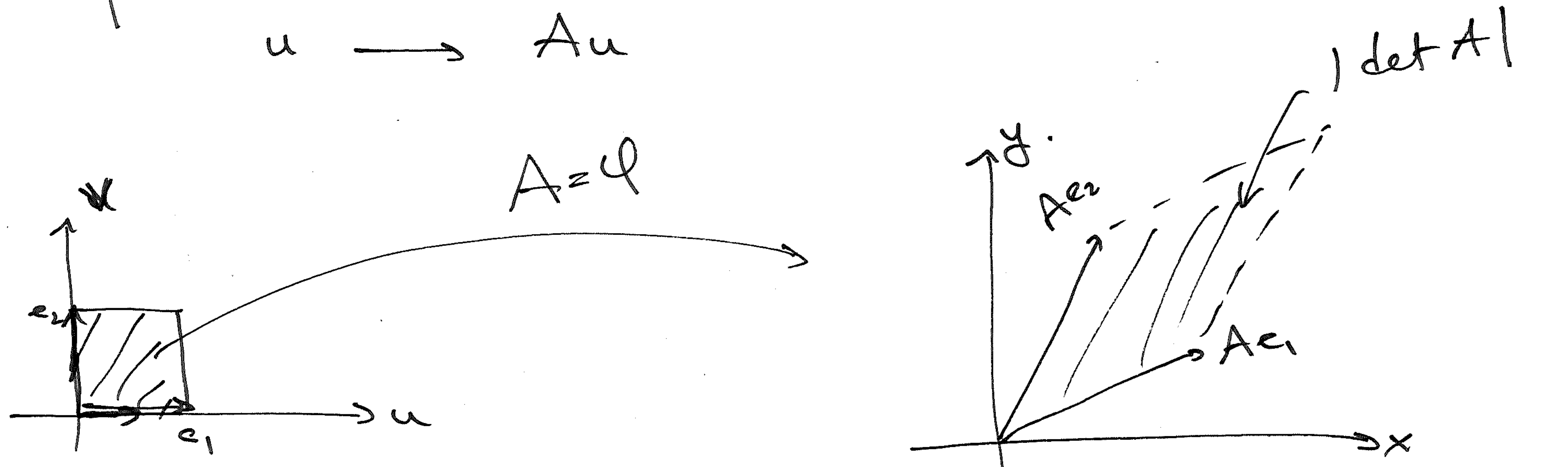
$$\int_{X=\varphi(U)} f(x)\, d\mu(x) = \int_U f(\varphi(u)) \left|\det \nabla\varphi(u)\right| d\mu(u).$$

$n=1$ $\quad |\det \nabla\varphi(u)| = |\varphi'(u)| = \operatorname{sgn}\varphi \; \varphi'(u)$

Why $\det(\mathrm{Jac})$ comes into this formula

Consider the linear map

$$\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad u \longrightarrow Au$$ where A is an $n \times n$ Matrix

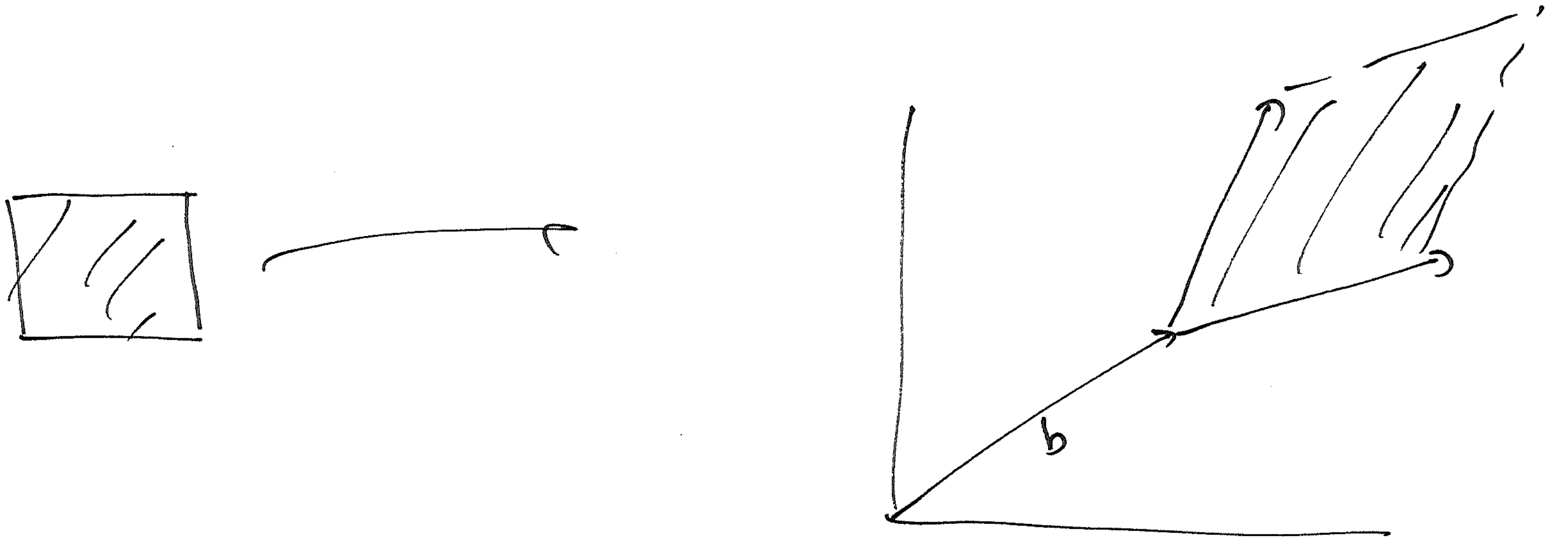


$1 \cdot du\,dv$.

$dx\,dy = |\det A|\, du\,dv$.

This is still the case if we had an affine linear map.

$\varphi(u) = Au + b$ for some fixed vector b.

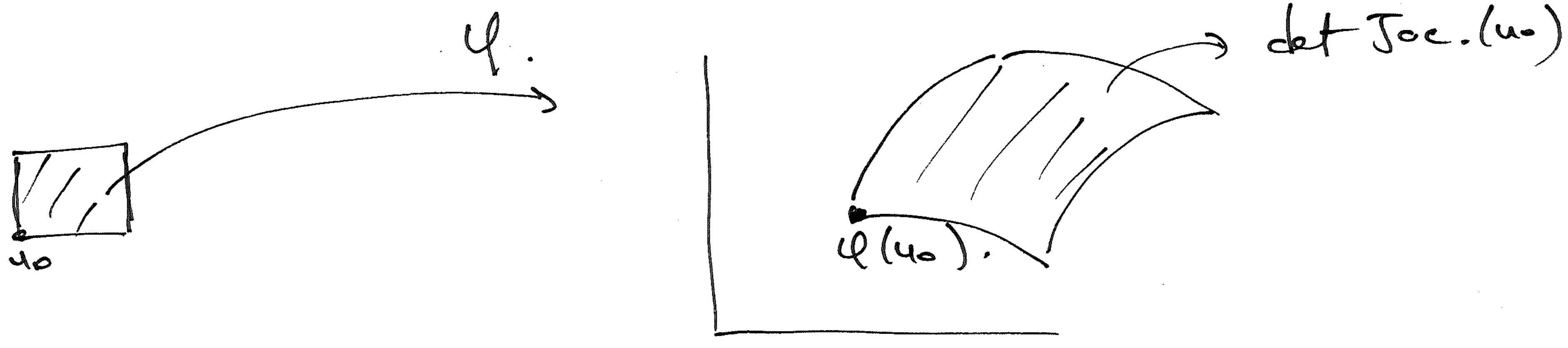


Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be non-linear transformation which is differentiable

Then at a point u near some fixed pt $u_0 \in \mathbb{R}^n$

$$\varphi(u) \approx \underbrace{\varphi(u_0) + \overbrace{d\varphi(u_0)}\,(u - u_0)}_{\text{affine linear map.}}$$

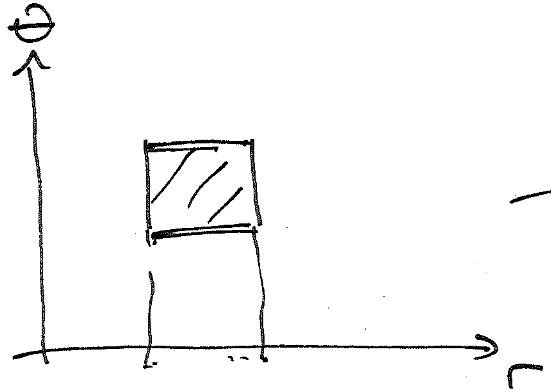
$b \rightsquigarrow \varphi(u_0) \qquad A \rightsquigarrow d\varphi(u_0) = \nabla\varphi(u_0) = (\mathrm{Jac}\,\varphi)(u_0)$



Recall $\nabla\varphi(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

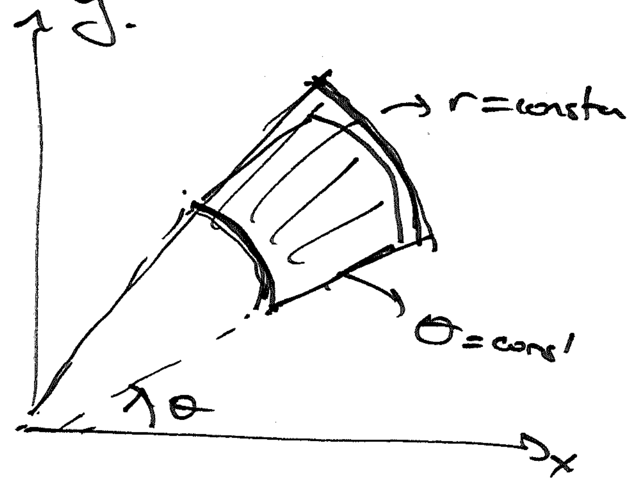
$\varphi = (\varphi_1, \varphi_2) = (x, y)$

① Polar coordinates.



$\varphi(r,\theta) = (x,y)$

$x = r\cos\theta$
$y = r\sin\theta$



$$\varphi(r,\theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

det

$$|d\varphi| = |\nabla\varphi| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

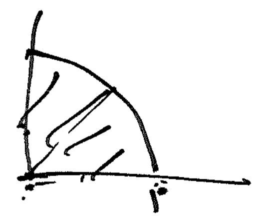
$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

det. of Jacobian.

Hence $dx\,dy = r\,dr\,d\theta$

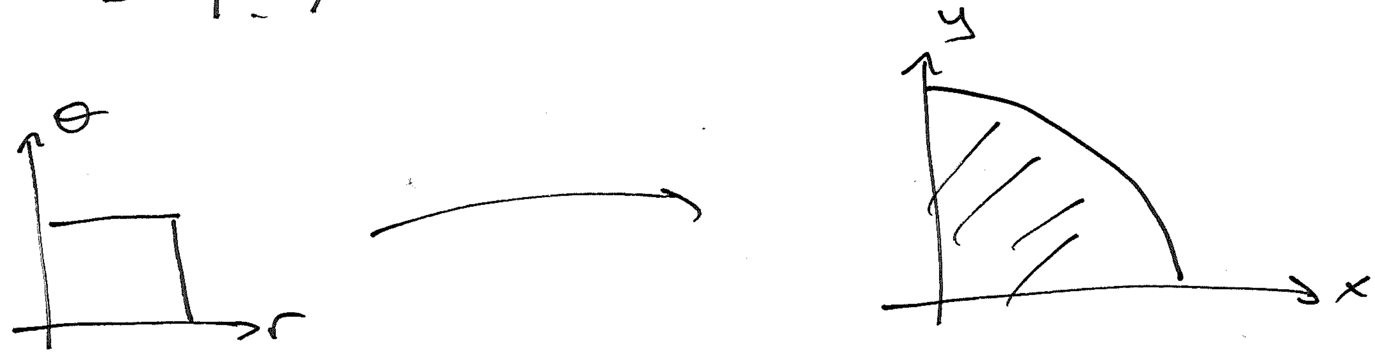
Ex. $\displaystyle\iint_D x\,dx\,dy$ where D is the quarter of the unit disc on the first quadrant



$$= \int_0^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx$$

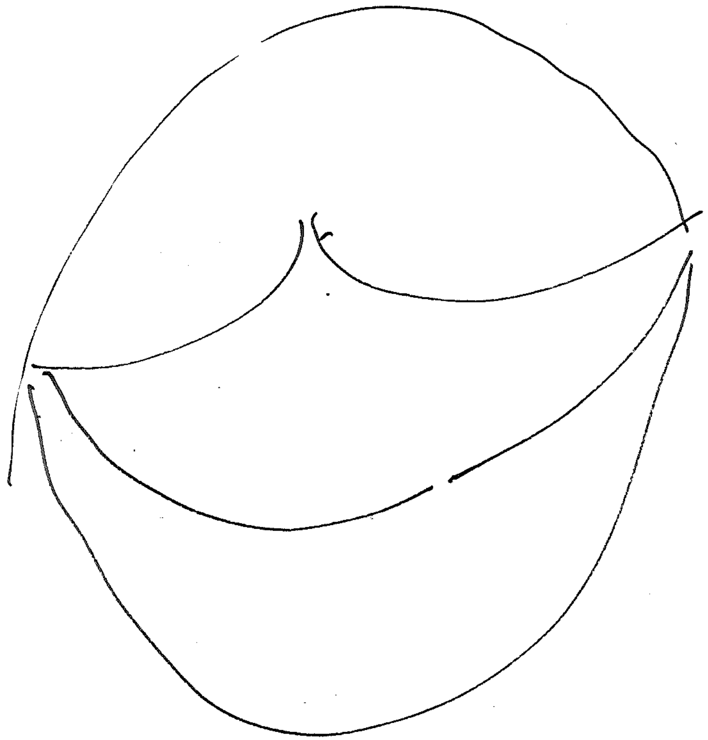
or

$$\int_{D=\varphi(U)} x\,dx\,dy = \int_0^{\pi/2}\int_0^1 r\cos\theta \; r\,dr\,d\theta$$



Example Find the volume between 2 surfaces

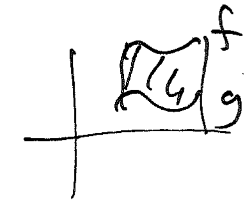
$z = x^2+y^2$ and $z = 50-x^2-y^2$.



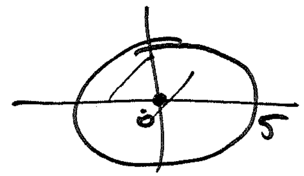
$z=x^2+y^2$ is a paraboloid open upwards

$z=50-x^2-y^2$ is " " open downwards

We first find their intersection



$\{(x,y) \mid x^2+y^2 = 50 - x^2 - y^2\}$.



$D = \{(x,y) \mid x^2+y^2 = 25\}$.

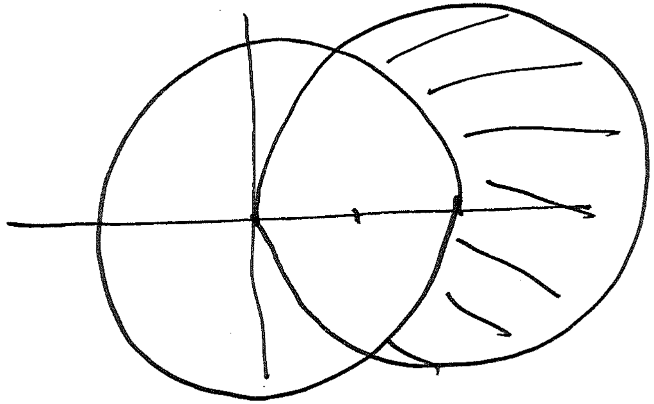
Hence the volume we are interested in

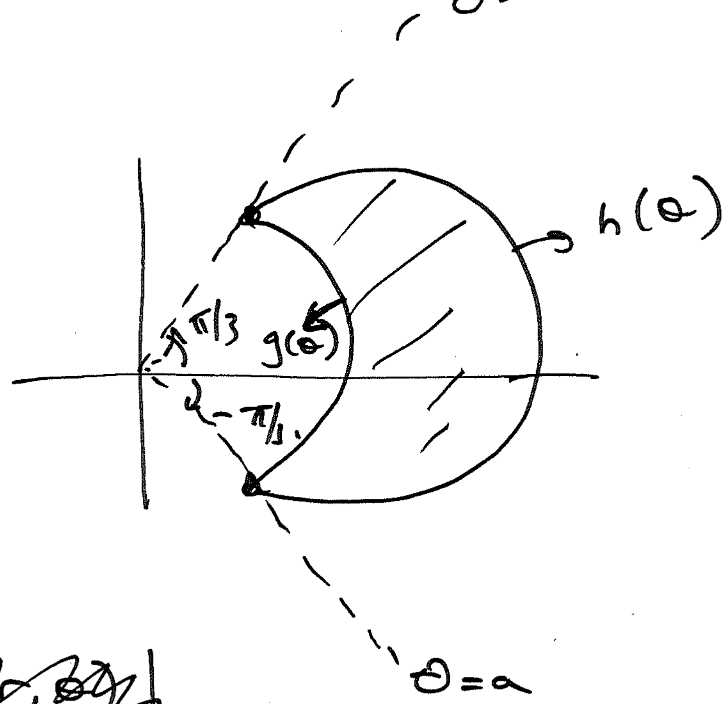
$$\iint_D 50 - x^2 - y^2 \, dx\, dy - \iint_D (x^2+y^2)\, dx\, dy.$$

$$= \underbrace{\iint_D \underbrace{50 - 2x^2 - 2y^2}\, \underbrace{dx\, dy}}_{} = 4 \int_0^1 \int_0^{\sqrt{25-x^2}} 50 - 2x^2 - 2y^2 \, dx\, dy.$$

$$\int_0^{2\pi} \int_0^5 (50 - 2r^2)\, \overbrace{r\, dr\, d\theta} = \cdots 625\pi.$$

Ex. Find the area outside the circle with radius 2, center (0,0), and inside the circle with center (2,0) and radius 2.





$$D = \{(r,\theta) \mid a < \theta < b, \ \underset{\substack{\| \\ 2}}{g(\theta)} < r \le h(\theta)\}.$$

$$\iint_D 1\,dx\,dy = \int_a^b \int_2^{h(\theta)} r\,dr\,d\theta.$$

$h(\theta)$ is on the circle $(x-2)^2+y^2=4$.

$$(r\cos\theta-2)^2+(r\sin\theta)^2=4.$$

$$r^2\cos^2\theta-4r\cos\theta+\cancel{4}+r^2\sin^2\theta=\cancel{4}.$$

$$r^2-4r\cos\theta=0$$

$$r(r-4\cos\theta)=0 \Rightarrow r-4\cos\theta=0 \Rightarrow r=4\cos\theta=h(\theta).$$

To find a, b we find the angle of intersection of $g(\theta)$, $h(\theta)$

$$g(\theta)=2=h(\theta)=4\cos\theta \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}, \frac{-\pi}{3}.$$

<u>Area</u> $= \displaystyle\int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3} \frac{r^2}{2}\Big|_2^{4\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3}(8\cos^2\theta-2)\,d\theta$

Using

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$2\cos^2\theta - 1 = \cos 2\theta.$$

$$8\cos^2\theta = 4(1 + \cos 2\theta)$$

$=$

$\vdots$

$$\frac{4\pi}{3} + 2\sqrt{3}.$$